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GR lecture notes of Neil Turok's lectures

Presentation · October 2020

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What is the scale on which gravity becomes very strong ?

when Newtonian potential $\frac{\Phi}{c^2} \sim 1$; dimensions $\frac{GM}{r} = v^2$

To probe a scale r, we need $\lambda < r$

$\Rightarrow E \sim \frac{hc}{\lambda} > \frac{hc}{r}$; $\frac{\Phi}{c^2} \sim \frac{GE}{c^4 r} > \frac{Gh\omega}{c^3 r^2} \sim 1 \because M = E/c^2$

for $r = \left(\frac{Gh}{c^3}\right)^{1/2} \equiv$ Planck length $\rightarrow$ smallest scale possible to probe geometry

ie. gravity will be very strong at scale r & we'll create a black hole

$$l_p = 1.6 \times 10^{-35} \text{ m}$$

largest scale ?

Dark energy is a component of energy that is constant in space & time ; even if the universe expands the energy density of dark energy remains fixed & this dark energy density ρ_Λ defines a scale.

$\frac{GM}{r} = G\rho_\Lambda r^2 \Rightarrow \frac{\Phi}{c^2} = \frac{G\rho_\Lambda r^2}{c^2}$ But $\frac{\rho_\Lambda}{c^2}$ is mass density

so $\frac{G\rho_\Lambda r^2}{c^4} \sim 1$ when $r = \left(\frac{c^4}{G\rho_\lambda}\right)^{1/2} = \frac{c^2}{\sqrt{G\rho_\lambda}}$

to make it dimensionless

ie. length scale defined by dark energy density $l_\lambda \sim \frac{c^2}{\sqrt{G\rho_\lambda}}$

(λ : cosmological constant which is Einstein's name for dark energy)

$= 17.4 \times 10^9$ lt yrs

$\sim 1.65 \times 10^{26}$ m

$\because$ Dark energy causes the expansion of universe to accelerate, it actually defines an upper limit scale beyond which we'll never see ; stuff that's beyond this length will accelerate from us very fast that we'll never see it, even if it emits light it'll never see us.

Let's take the geometric mean $\sqrt{l_p l_\lambda} \simeq 10^{-4} m$ (scale of living cells)

- Maxwell's equations:

$\vec{\nabla}.\vec{E} = \frac{\rho_e}{\varepsilon_0}$ Gauss's law

$\vec{\nabla}.\vec{B} = 0$ No monopole

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla}\times\vec{B} = \mu_0 J_e + \boxed{\mu_0\varepsilon_0\frac{\partial E}{\partial t}}$ Ampere's law

"displacement current" ← without this term Maxwell's eqn was inconsistent

← it's this term that gave rise to electromagnetic waves travelling at speed of light

$\mu_0 \equiv 4\pi \times 10^{-7}\ N/A^2$

$\varepsilon_0 \approx \frac{1}{4\pi \times 9 \times 10^9}\ C^2/Nm^2$

} are quantities that were observed experimentally by applying Gauss's law & Ampere's law

Recall: $\vec{\nabla}.\vec{E} = \partial_i E_i$ $i = 1,2,3$; Einstein's summat^n convention (repeated indices are summed)

$(\vec{\nabla}\times\vec{E})_i = \epsilon_{ijk}\partial_j E_k$; $\partial_i = \frac{\partial}{\partial x^i}$, $x^i = (x,y,z)$

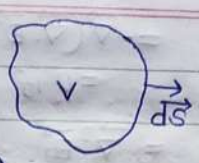
ϵ_{ijk} is totally antisymmetric, $\epsilon_{123} = +1$

Let's check Maxwell's eq^n for consistency.

from definitions of ρ_e, $\vec{J}_e$ we have

$$\frac{\partial \rho_e}{\partial t} + \vec{\nabla}.\vec{J}_e = 0 \quad \text{--- continuity eq}^n$$

This we'll get by taking time derivative of $\mathcal{Q}$ and equating it with 2nd eq^n & then using divergence theorem



$$\mathcal{Q} = \int_V d^3x\, \rho_e(x)$$

$\vec{J}_e$ = electric current density

$$\frac{d\mathcal{Q}}{dt} = -\int_S \vec{J}_e \cdot \vec{ds}$$

consider Faraday's law

$$\vec{\nabla}.(\vec{\nabla}\times\vec{E}) = \partial_i\, \varepsilon_{ijk}\, \partial_j E_k = 0 \rightarrow$$

& $\vec{\nabla}.\left(\frac{\partial \vec{B}}{\partial t}\right) = 0$ --- by no monopoles law

$\rightarrow$ partial

$\because$ i) derivatives commute $\Rightarrow \partial_i \partial_j = \partial_j \partial_i$ ie. derivative term is symmetrical under i & j but ε_{ijk} is antisymmetrical

ii) contraction of a symmetric thing with an antisymmetric thing gives zero.

consider Ampere's law

$\vec{\nabla}.(\vec{\nabla}\times\vec{B}) = 0$ --- by same arguement as above

$\mu_0 \vec{\nabla}.\vec{J}_e \neq 0$ if $\frac{\partial \rho_e}{\partial t} \neq 0$

Maxwell noticed that if you add displacement current

$$\mu_0 \varepsilon_0 \frac{\partial \vec{\nabla}.\vec{E}}{\partial t} \xrightarrow{\text{gauss}} \mu_0 \frac{\partial \rho_e}{\partial t}$$

$$\Rightarrow \mu_0\left(\vec{\nabla}.\vec{J}_e + \frac{\partial \rho_e}{\partial t}\right) = 0$$ ie. we can make eq^n consistent

Amazing consequence: Even if we set ρ_e & $\vec{J}_e$ to be zero, there are still very interesting solutions

To see that, take time derivative of 4th eq^n,

$$\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times (\vec{\nabla} \times \vec{E})$$

$$= -\vec{\nabla}(\vec{\nabla}.\vec{E}) + \vec{\nabla}^2 \vec{E} \quad \text{------} \quad \varepsilon_{ijk}\varepsilon_{nlm}\partial_j \partial_l E_m$$

$$= 0 \text{ if } \rho_e = 0 \qquad \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$= \vec{\nabla}^2 \vec{E}$$

ie. wave equation for $\vec{E}$ with $c^2 = \frac{1}{\mu_0 \varepsilon_0} \simeq 3 \times 10^8$ m/s

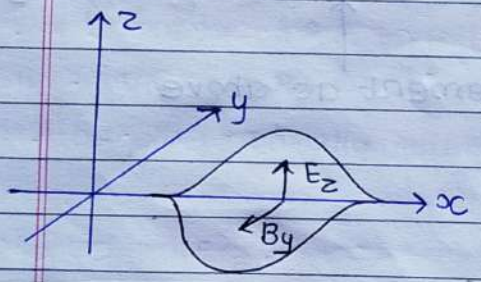
In general, wave eqn is $\left(\frac{\partial^2}{\partial t^2} + c^2 \vec{\nabla}^2\right) f = 0$

↑ conversion factor between space & time here

Example : $\vec{E} = (0, 0, f(x-ct))$ we are taking only z-component of E to be non-zero, because if a wave is travelling along x direction, then by we got to satisfy div. $E = 0$ ie. $\frac{\partial}{\partial x} E_x = 0$; easy way to satisfy that is to set $E_x = 0$

$\vec{B} = \left(0, -\frac{1}{c} f(x-ct), 0\right)$

↑ c is used as conversion factor, because from Fraday's law on LHS we have space derivative & on RHS we have time derivative



← linearly polarized wave

fact is we get $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ which is completely inconsistent with Galileo + Newton's understanding of mechanics (ie. if two objects are moving relative to each other, then speeds change)

conformal symmetry : when your length scales, theories remain invariant

- Symmetries in Maxwell's theory :

Newton & Galileo assumed that if two objects are moving and one catches up the other, then according to second object the first one is moving slowly & when we trace them back then there comes an invariance in Newton's equations called Galilean invariance (that one can change its reference frame by moving to a frame that has constant velocity & law remains same ie. Newtons laws hold in all frames related to one another by a Galilean transformation) ie. all frames related to one another by such a transformation are inertial.; also called Newtonian relativity.

· Axioms from Newton's theory :

1. There exists an absolute space, in which Newton's law are true. An inertial frame is a reference frame in relative uniform motion to absolute space.
2. All inertial frames share a universal time.

- Galilean invariance is no longer true in Maxwell's theory ; it's replaced by another invariance which preserves speed of light

- M.E's has a lot of symmetries ; translation invariance, Lorentz invariance, conformal invariance (fundamental ; includes scale invariance)

connect space & time

- In Lorentz symmetry, we combine space & time $x^\alpha = (ct, \vec{x})$ ← to make dimensions same

$$\partial^\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad ; \alpha = 0, 1, 2, 3$$

metric $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ '-1' plays a very imp. role; infact the only difference between space & time is this negative sign; the role

η is to put -ve sign in time direction which is imp. in order to get correct Maxwell's eqn; this -ve sign is responsible for the causal structure of physics; it changes the eqn of physics from parabolic (having all diagonal elements to be +1) to hyperbolic (we have initial value problems & we can propogate things forward in tim

inverse $\eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

so $\eta_{\alpha\beta}\eta^{\beta\delta} = \delta_\alpha^{\ \delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (row ↑, column →)

wave operator $\eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2$; $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta = -\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}$

↑ Here we contract indices with one up & other down which will ensure that the quantity will be invariant under symmetry

Simplified M.E's :

$\vec{E}, \vec{B}$ ↑ 6 components

if we introduce an object $F^{\alpha\beta}$ ← 16 components but what we can do is to introduce antisymmetry

$$F^{\alpha\beta} = -F^{\beta\alpha}$$

diagonal elements will be zero due to antisymmetry

$$\begin{pmatrix} 0 & \times & \times & \times \\ -(\ ) & 0 & \times & \times \\ & & 0 & \times \\ & & & 0 \end{pmatrix}$$

← non-zero components

here we'll have just -ve of non-zero components

} in this way we'll have 6 ind. components of an antisymmetric 2index tensor ie. rank 2 antisymmetric tensor

$$F_{\alpha\beta} = -F_{\beta\alpha} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

i.e. $F_{0i} = -\frac{E_i}{c}$; $F_{ij} = \epsilon_{ijk} B_k$

Also $F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ & 0 & B_z & -B_y \\ - & & 0 & B_x \\ & & & 0 \end{pmatrix}$

four-current $J^\alpha = (\rho_e c, \vec{J})$; turns out M.E.'s can be written

$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$ ---- ① → when $\beta = 0$; Gauss's law
→ $\beta = i$; Ampere's law

$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0$ ---- ②

↳ no monopole & Faraday's law because we don't have current density on RHS

we can further simplify ② introducing $\epsilon^{\alpha\beta\gamma\delta}$

(ϵ^{ijk} & δ_{ij} are the only invariant tensors under rotation)

$\epsilon^{\alpha\beta\gamma\delta} = 1$; $\alpha\beta\gamma\delta$ is even permutat^n of 0,1,2,3 (how many pair flips)
$= -1$; 'odd'
$= 0$; otherwise

so (2) becomes $\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$ ---- (By picking $\alpha = 0$, $\beta = 1, \gamma = 2, \delta = 3$)

if we pick $\alpha = 0 \Rightarrow$ no monopoles

ie. for $\alpha = 0$ $\epsilon^{\alpha\beta\gamma\delta}$ will only involve spatial derivatives

we'll get

$$\epsilon^{ijk} \partial_i F_{jk} = 0$$

$$\partial_i B_i = 0 \quad \because F_{jk} = \epsilon_{ijk} B_i$$

no

$\Rightarrow$ no mopoles

for $\alpha = i$, we'll get Faraday's law

Having reduced these eq^n, we can derive waves that satisfy these eq^n

Taking divergence of (2); ∂^α (2) and putting $J^\beta = 0$

so $\partial^\alpha \partial_\alpha F_{\beta\gamma} + \partial^\alpha \partial_\gamma F_{\alpha\beta} + \partial^\alpha \partial_\beta F_{\gamma\alpha} = 0$

($\partial_\gamma F_{\alpha\beta} = 0$, $\partial_\beta F_{\gamma\alpha} = 0$)

$\Rightarrow \partial^\alpha \partial_\alpha F_{\beta\gamma} = 0$ ------wave equation

Actually, $\varepsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$ $\Rightarrow$ Poincare's Lemma which states that $F_{\alpha\beta}$ can be written interms of a vector potential

$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$

or $F_{\gamma\delta} = \partial_\gamma A_\delta - \partial_\delta A_\gamma$; so this should satisfy above eq^n,

$\varepsilon^{\alpha\beta\gamma\delta} \partial_\beta \partial_\gamma A_\delta - \varepsilon^{\alpha\beta\gamma\delta} \partial_\beta \partial_\delta A_\gamma = 0$

($\varepsilon^{\alpha\beta\gamma\delta}$: antisymmetric; $\partial_\beta \partial_\gamma$: symmetric; $\partial_\beta \partial_\delta A_\gamma$: 0)

ie. Poincare showed that if we have an eq^n like above (which implies that curl of a quantity is zero), then that quantity can be written in terms of gradient.

$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is only true locally (if spacetime is not a simple manifold; if it has, say a hole in it then we've to be careful with this eq^n)

or $F_{\gamma\delta} = \partial_\gamma A_\delta - \partial_\delta A_\gamma$

A_δ ; vector potential $\rightarrow$ it allows us to write a field tenson $F_{\alpha\beta}$ in terms of a vector potential

set $A^\mu = \left(\frac{\Phi}{c}, \vec{A}\right)$, $A_\mu = \left(-\frac{\Phi}{c}, \vec{A}\right)$

$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$

(usual expressions in electrodynamics)

- Notice that A^μ four-vector potential is not unique.
If we replace $A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \chi$

χ : any function

then $F_{\mu\nu} \longrightarrow F_{\mu\nu}' = F_{\mu\nu} + \partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi = F_{\mu\nu}$

∵ partial derivatives commute

$\Rightarrow$ if we replace vector potential with another one, defined as above, then we get the same field strength; so if the only physical thing in the theory is field strength (that's the way Maxwell defined it, purely in terms of field strength), then A_μ has some redundancy $\longrightarrow$ Gauge invariance, another symmetry of M.E.'s!

- so M.E.'s has translation invariance, which is quite obvious (if we perform experiments on diff. place or some another day, then its the same), Lorentz invariance (generalization of rotations), scale invariance (one can rescale all lengths)

↑

If one asks what were the fundamental laws that governed physics (in 20th century), these were it, & they were the seeds for all 20th century physics; of course there was GM, but that also arised because of inconsistency between Maxwell's theory & theory of heat.

- Lorentz invariance

we've reduced electrodynamics to

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$$

$$\varepsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

These (deliberately) are tensor equations, constructed by contracting indices in a very specific way (upper indices with lower)

↑ summing over two indices

Let's compare with classical mechanics

$$m\ddot{\vec{x}} = -\vec{\nabla}V$$ --- invariant under rotations

ie. $\vec{x} \to \vec{x}' = O\vec{x}$ where $O^T O = 1$ (orthogonal matrix)

$$\Rightarrow m\ddot{\vec{x}}' = -\vec{\nabla}'V$$

$$\text{so } -\vec{\nabla}'V = -(O^{-1})^T \vec{\nabla}V = -O\vec{\nabla}V$$

ie. if we rotate a system, it still satisfies the same eqn ; in order for that to be true LHS & RHS should be a tensor ie. they should have same no. of indices on both sides to have a sensible eqn & these eqn have to transform in a same way under rotation.

- In electrodynamics, we can "rotate" spacetime coordinates

$$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta \quad ; \Lambda \text{ is a } 4\times4 \text{ matrix}$$

$$\Rightarrow \partial_\alpha = \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta} \quad \text{---chain rule of differention}$$

$$\partial_\alpha = \Lambda^\beta{}_\alpha \partial'_\beta$$

(Remember coordinates have upper index, derivatives have lower index)

this is how derivatives transform

$$\Rightarrow \partial'_\alpha = \tilde{\Lambda}^\beta{}_\alpha \partial_\beta$$

; $\tilde{\Lambda}$ is the inverse of Λ ; we define inverse (twiddle) to satisfy this eqn

$$\Lambda^\alpha{}_\beta \tilde{\Lambda}^\beta{}_\gamma = \delta^\alpha{}_\gamma = \Lambda^\beta{}_\gamma \tilde{\Lambda}^\alpha{}_\beta$$

if we multiply on the right with $\tilde{\Lambda}$, we get identity matrix, or we multiply on the left, we get identity matrix

- so coordinates transform using Λ & derivatives transform using $\tilde{\Lambda}$; in general, upstair quantities will go like Λ & downstairs quantities will go like $\tilde{\Lambda}$

- Because Λ & $\tilde{\Lambda}$ are inverses, Λ's will disappear & eqn will be invariant

four-vectors with upstairs indices (contravariant vectors)

$$J'^\alpha = \Lambda^\alpha{}_\beta J^\beta$$

trees grow up

four vectors with downstairs indices (covariant vectors)

$$\partial'_\alpha = \tilde{\Lambda}^\beta{}_\alpha \partial_\beta$$

values go down,

so, $$F'^{\alpha\beta} = \Lambda^\alpha_\gamma \Lambda^\beta_\delta F^{\gamma\delta}$$

$$\varepsilon'^{\alpha\beta\gamma\delta} = \varepsilon^{\chi\psi\eta\xi} \Lambda^\alpha_\chi \Lambda^\beta_\psi \Lambda^\gamma_\eta \Lambda^\delta_\xi$$

$$= \varepsilon^{\alpha\beta\gamma\delta} \det\Lambda$$

(under point 'simplified M.E.'s)

Most important : η is used to define $F_{\alpha\beta}$ in terms of $F^{\alpha\beta}$. so we must insist that η is invariant

ie. $$\boxed{\eta'^{\alpha\beta} = \Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta^{\gamma\delta} = \eta^{\alpha\beta}}$$ ----- defining eqn of Lorentz transformations

has to be a matrix which preserves $\eta^{\alpha\beta}$

$$\Rightarrow \Lambda \eta \Lambda^T = \eta$$

$$\det\Lambda \det\Lambda^T \det\eta = \det\eta$$

$$\det\Lambda \det\Lambda = 1$$

$$\Rightarrow (\det\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$$ ie. L.T. split into two pieces,

those with $\det\Lambda = 1$ (includes identity transformation $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta$) &

those with $\det\Lambda = -1$

↳ "connected component" ie. its smoothly connected to the identity

Now we have discovered this huge symmetry in M.E.'s under which that

$$(ct, x)' = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix}$$

eg. Let take Λ to only have identity at the lower corner ie. it only mixes x with t

$$\left(\begin{array}{c|cc} M & & \\ \hline & 1 & 0 \\ & 0 & 1 \end{array}\right)$$ then we must have $M \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

↑ if this = 1 then its an orthogonal matrix

orthogonal matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

↑

ie. it satisfies $O^T O = 1$

; only difference the minus sign makes is we get $M = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$

ie. $\begin{pmatrix} c & s \\ s & c \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & s \\ s & c \end{pmatrix} = \begin{pmatrix} -c^2+s^2 & 0 \\ 0 & c^2-s^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

---Lorentz transformations

(responsible for all the effects in SR)

$ct' = \cosh\theta\, ct + \sinh\theta\, x = \gamma(ct + \frac{v}{c}x)$ where $\sinh\theta = \gamma\frac{v}{c}$

$x' = \cosh x + \sinh\theta\, ct = \gamma(x + vt)$ $\cosh\theta = \gamma$

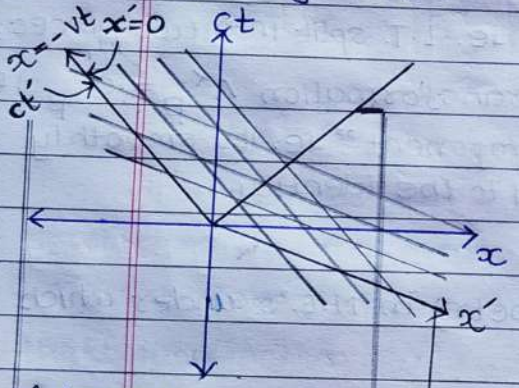
$\Rightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

What's the interpretation of this Lorentz transformations?

$ct' = \cosh\theta\, ct + \sinh\theta\, x = \gamma(ct + \frac{v}{c}x) = \cosh\theta\,(ct + \tanh\theta\, x)$

what does this $\tanh\theta$ mean?

so we actually traded one parameter v/c ; $v < c$ θ (which we really don't what it meant physically) into 'v'



At which value of x (at each t) would our x' not be moving? This point, by definition, would be the origin in x' system.

$x' = 0$ for $x = -vt$

locus of points which will have $x' = 0$

locus of points where $ct' = 0$ ie. $ct = -\frac{v}{c}x$

what L.T does is to keep this line constant; so if we take x & ct, open them or close them, the line remains invariant

- Interpretation of 'v': if we have an observer 'O' at (ct, x) & another new observer uses (ct', x') coordinates to observe, then this observer is moving at a speed $-v$ in the original (ct, x) coordinate system, because new observers locus in original coordinate system is $x = -vt$

- so its rather straightforward to go from these abstract angles to the physical interpretation of the coordinates, which is that new observer is moving relative to the original one

Why do we impose a condition for η to be always invariant ?

Demanding η to be the same in any coordinate system is the same as demanding Lorentz invariance

- η has to be the same because we want our equations to be valid under any coordinate system ; if η transformed, we don't have any eq^n to determine η, so we don't want η to transform

- η is the metric on spacetime (quantity we use to measure distances on spacetime) is insisted to be invariant under change of coordinates ; same is with rotations, metric in spacetime is actually the pythagoras rule (distance bet^n two points) ; rotations can be defined as set of operations which do not change lengths & η defines a length on spacetime

$\bullet x_1^\alpha$ $\quad \bullet x_2^\alpha$ ← Two points, with seperation $x_1^\alpha - x_2^\alpha$ in spacetime

we want to define a distance, we generalize pythagoras

$$\eta_{\alpha\beta}(x_1^\alpha - x_2^\alpha)(x_1^\beta - x_2^\beta) = \text{Lorentz invariant distance}$$

that's where the word metric comes from i.e. measuring distance (in meters) ; η gives us a way of defining distances that are Lorentz invariant, its insisting that this metric (way we measure distances) is the same, no matter which coordinate system we use

- Also when we construct M.E.'s we needed η to get the signs right & insisting that η is always the same is what forces us to Lorentz transformations

Maxwell's eq^n $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$ ——(1)
$\varepsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0$ ——(2)
} Do these eq^n follow from variational principle ?

Notice from (1), ∂_ν(1) is $\partial_\mu\partial_\nu F^{\mu\nu} = -\mu_0 \cdot \partial_\nu J^\nu \Rightarrow$ current is conserved
$\partial_\mu\partial_\nu F^{\mu\nu} = 0$
derivatives commute (symmetrical) & $F^{\mu\nu}$ is antisymmetric

(♣)

ie. we want to write an action whose variat^n yields these eq^n.

Set $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow$ automatically satisfy (2) ; it turns out

action will be

$$S = \int \underbrace{d^4x}_{c\,dt\,d^3x} \left(-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J^\alpha A_\alpha \right)$$

volume element of spacetime ; it's a scalar under Lorentz transformation

proof ↓

$$\Rightarrow d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x \quad \text{and} \quad x' = \Lambda x \Rightarrow \frac{\partial x'}{\partial x} = \Lambda$$

↑ Jacobian

$$d^4x' = \underbrace{\det\Lambda}_{+1}\, d^4x$$

so d^4x is Lorentz invariant, also $F_{\mu\nu}$ is Lorentz invariant & J is always conserved, we have as a whole an invariant quantity ; that's why action is very imp. because it incorporates all the symmetries in the theory

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

∵ the action is going to be Lorentz invariant, so η will be invariant & everything else has to transform according to Lorentz transformat^n

Let's vary action w.r.t A_μ

$$\delta S = S(A_\mu + \delta A_\mu) - S(A_\mu) \quad \text{to 1st order in } \delta A_\mu$$

$-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$ ↑

these two quantities are same, so we'll get square, after differentiating we get a factor of 2 ; also there's two quantities so there be 2 again

$(\partial_\mu A_\nu - \partial_\nu A_\mu)\ \eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$

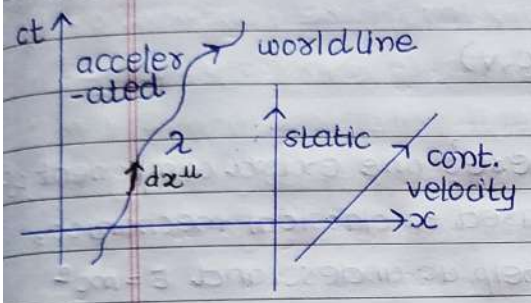
$$\delta S = \int \left(-\frac{1}{\mu_0} \partial_\mu \delta A_\nu F^{\mu\nu} + J^\alpha \delta A_\alpha \right) d^4x = 0$$

integrating by parts $= +\frac{1}{\mu_0} \delta A_\nu \partial_\mu F^{\mu\nu}$

$$\delta S = \int d^4x \left(\frac{1}{\mu_0} \partial_\mu F^{\mu\nu} + J^\nu \right) \delta A_\nu = 0 \qquad \forall \delta A_\nu$$

$$\Rightarrow \frac{1}{\mu_0} \partial_\mu F^{\mu\nu} + J^\nu = 0$$

Particles :



λ will be a good parameter if it varies monotonically along the line, so that every point on line will have a unique value

$$x^\alpha(\lambda) = (ct, x)(\lambda)$$

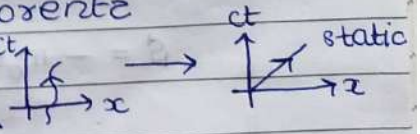
Lorentz invariant length of dx^μ

$$\eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + d\vec{x}^2 \begin{cases} < 0 & \text{curve is timelike at that point} \\ = 0 & \text{null (no Lorentz invariant length)} \\ > 0 & \text{spacelike} \end{cases}$$

- Massive particles follow timelike worldlines ; massless particles (photons, gravitons) follow null worldlines & tachyons (fictitious particles, doesn't exist) would follow spacelike worldlines

- Let's define τ = proper time along the world line for massive particles via $d\tau = \sqrt{dt^2 - d\vec{x}^2/c^2} = \sqrt{-\eta_{\mu\nu} \frac{dx^\mu dx^\nu}{c^2}}$ which is Lorentz invariant

ie. it will be equal to dt in a Lorentz frame where $d\vec{x} = 0$

ie. if we take a particle worldline & do a Lorentz transformation to make that particle static 

then $d\tau = dt$ in lorentz frame where particle is static ($d\vec{x} = 0$)

ie. if we have a clock with this particle, then reading of clock will be $d\tau$ ie. time measured in a frame where particle is static

In general frame, $d\tau = dt\sqrt{1-\vec{v}^2/c^2}$ where $\vec{v} = \frac{d\vec{x}}{dt}$ is the velocity of particle

$$dt \equiv \frac{dt}{\gamma} \quad ; \gamma = \frac{1}{\sqrt{1-\vec{v}^2/c^2}}$$

Total proper time along the particle worldline, $\tau = \int d\tau = \int \frac{dt}{\gamma}$

4-velocity of particle, $U^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, v)$

4-momentum $m_0 U^\mu = p^\mu$ ..... Here, there's one extra component of p compared to classical mechanics, that'll help us understand $E = mc^2$ because its time component is related to energy

~~rest mass~~ (~~doesn't change with L.T.~~)

Now we want to S that is Lorentz invariant, what invariant quantity could we possibly use? → total length of the line

Action for particle $S = \int \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$

This doesn't have right dimensions for action

c^2dt^2 hence factor $1/c$

what are dim. of action $\frac{i}{\hbar}\int dt\, E = px$ ie. dim. of action is Et

dim. of Lagrangian

dim. of Et

will give quantum amplitude

~~Planck in his paper said that he's going to introduce this constant which is 'quantum of action'~~

- Now we've multiply S with something that has dim. of energy (ie. $\frac{1}{2}mv^2$)

$$S = -m_0c^2\int \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} = -m_0c\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

Action is 'reparametrizat^n invariant'

ie. $\lambda \to \lambda'(\lambda)$ then $\frac{d\lambda'}{d\lambda} \neq 0 \Rightarrow$ if λ changes by $d\lambda$ then λ' also changes

so we can write $S = -m_0 c \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$

which also means that we can pick λ to be anything we like ; in particular, we can pick λ to be t (it's a good parameter)

$$S = -m_0 c \int c\,dt \sqrt{1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2} \approx \int dt \left(-m_0 c^2 + \frac{1}{2} m_0 \left(\frac{d\vec{x}}{dt}\right)^2 + \dots\right)$$

($-m_0 c^2$ is just a constant if we vary S, it's not gonna change) But it looks like P.E. $= m_0 c^2$ clue for $E = mc^2$

($\frac{1}{2} m_0 \left(\frac{d\vec{x}}{dt}\right)^2$: Newtonian K.E.)

- The thought experiment by Einstein to prove $E = mc^2$ was quite unnecessary ; It was Noether who actually told us what energy is.

$E = mc^2$ is properly explained by Noether's theorem :

" for every continous (global) symmetry of an action (up to surface terms), there's a conservation law "

Any continous symmetry has an infinitesimal version

$$q^a(t) \longrightarrow q^{a'}(t) = q^{(a)}t + \varepsilon^{ab} \overset{\circledast}{x_b}(q, \dot{q}, t)$$

(ε^{ab} : (small) parameters of symmetry ; x_b : generators of symmetry)

$S = \int dt\, L(q, \dot{q}, t)$ $\quad \circledast$ is a symmetry if $dS = 0$ for cons. ε

$$\delta S = S' - S = \int dt \left(\dot{\varepsilon}^{ab} J_{ab}(q, \dot{q}, t) + \varepsilon^{ab} \frac{d}{dt} K_{ab}(q, \dot{q}, t)\right)$$

Noether's trick :
Let ε^{ab} be a funct^n of time t; so we'll get zero when ε^{ab} is not funct^n of t.

Lagrangian involves only 1st derivatives of q, so action can only involve ε & $\dot{\varepsilon}$, so it must be linear in ε & $\dot{\varepsilon}$

these are the only terms consistent with $\circledast$ being a global symmetry of the action

Integrating by parts,

$$\delta S = \int dt\ \varepsilon^{ab} \frac{d}{dt} (K_{ab} - J_{ab})$$

Now we have this eqn which is valid when ϵ is any funct^n of time ; so we can regard ϵ^{ab} differently ie. we can regard ϵ^{ab} as variation of action

$\Rightarrow$ (*) is a legitimate variation, for any $\epsilon^{ab}(t)$, so

$\delta S = 0 \iff$ E.O.M's are satisfied

$\Rightarrow$ if E.O.M's are satisfied, $\frac{d}{dt}(k_{ab} - J_{ab}) = 0$ ---- conservation law

- So this is what Noether tells us, you take an action which has a symmetry, you allow the symmetry parameters to be functions of time you vary the action, you integrate by parts to bring symmetry parameter outside & you say that it is a legitimate variation & it must be that the variation vanishes if E.O.M's are satisfied (for any ϵ^{ab}) & therefore this coefficient $\frac{d}{dt}(k_{ab} - J_{ab})$ vanishes & we get conservat^n law

for eg. ① free Newtonian particle $S = \int \frac{1}{2} m \dot{\vec{x}}^2 \, dt$ $\quad \vec{x} \to \vec{x} + \epsilon(t)$ translat^n invariance

$\delta \vec{x} = \epsilon(t)$

$\delta \dot{\vec{x}} = \dot{\vec{\epsilon}}(t)$

$$\delta S = \int m \dot{\vec{x}} \, \delta \dot{\vec{x}}$$
$$= \int m \dot{\vec{x}} \, \dot{\vec{\epsilon}}(t)$$
$$= -\int (m \dot{\vec{x}})^{\cdot} \vec{\epsilon}$$
$$= 0 \qquad \forall \ \vec{\epsilon}(t)$$

$\Rightarrow \frac{d}{dt}(m \dot{\vec{x}}) = 0$ --- so that's how we get momentum explicitly using Noether's theorem

② particle in a potential $S = \int \left(\frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) \right) dt$ V depends on x, so S is not translat^n invariant ; but its time invariant

Time translat^n $\vec{x}(t) \longrightarrow \vec{x}(t+\epsilon) \approx \vec{x}(t) + \epsilon \dot{\vec{x}}(t)$

time translation has a generator that depends on velocity

$\delta \vec{x} = \epsilon \dot{\vec{x}}(t)$; $\delta \dot{\vec{x}} = (\epsilon \dot{\vec{x}}(t))^{\cdot}$

$\therefore \delta S = \int m \dot{\vec{x}} (\epsilon \dot{\vec{x}}(t))^{\cdot} - \vec{\nabla} V \, \epsilon \dot{\vec{x}}(t)$

$$= \int -(m\ddot{\vec{x}})\cdot\varepsilon\dot{\vec{x}} - \vec{\nabla}V\cdot\varepsilon\dot{\vec{x}} \quad \text{---- integrating by parts}$$

$$= -\int \varepsilon \frac{d}{dt}\left(\frac{m\dot{\vec{x}}^2}{2} + V(\vec{x})\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{\vec{x}}^2 + V(\vec{x})\right) = 0 \quad \text{---- energy conservation}$$

In general, $S = \int dt\, \mathcal{L}(q, \dot{q}, t)$

If S is translation invariant, then $q^a \to q^a + \varepsilon^a$ is a symmetry

$$\delta S = \int \left(\underbrace{\frac{\partial L}{\partial q^a}}_{0 \text{ by translation invariance}}\varepsilon^a + \frac{\partial L}{\partial \dot{q}^a}\dot{\varepsilon}^a\right) \Rightarrow p_a = \frac{\partial L}{\partial \dot{q}^a} \text{ is conserved}$$

S is time invariant if $\frac{\partial L}{\partial t} = 0$

$$q^a \to q^a + \varepsilon\dot{q}^a$$
$$\dot{q}^a \to \dot{q}^a + (\varepsilon\dot{q}^a)\dot{}$$

$$\delta S = \int \left(\frac{\partial L}{\partial q^a}\varepsilon\dot{q}^a + \frac{\partial L}{\partial \dot{q}^a}(\varepsilon\dot{q}^a)\dot{}\right)$$

$$= \int \varepsilon\left[\frac{\partial L}{\partial q^a}\dot{q}^a - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)\dot{q}^a\right] \quad \text{... int. by parts}$$

$$\frac{d}{dt}\left[L - \frac{\partial L}{\partial \dot{q}^a}\dot{q}^a\right] = \frac{\partial L}{\partial q^a}\dot{q}^a + \cancel{\frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)\dot{q}^a - \cancel{\frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a}$$

$$\Rightarrow \delta S = \int \varepsilon \frac{d}{dt}\left[L - \frac{\partial L}{\partial \dot{q}^a}\dot{q}^a\right] \Rightarrow \frac{d}{dt}(L - p_a\dot{q}_a) = \frac{d}{dt}\underbrace{(p_a\dot{q}^a - L)}_{H} = 0$$

so Hamiltonian is conserved if the Lagrangian doesn't have explicit time dependence

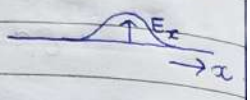
- what's imp. in this arguement $S = \int dt\, \mathcal{L}(q, \dot{q}, t)$ is that we had a time, one preferred time & if we start changing this time, it's not clear how this symmetry applies ; so some of the lorentz transformation it does apply to ie. for rotations ; but boosts mix space & time & they don't give rise to conservation laws (because they change time itself)

- But then electric charge is also conserved, what's the symmetry there then?

→ symmetry is the phase of the field that carries the electric charge. $\psi \to e^{i\phi}\psi$ --- probability is the same because ϕ doesn't come in $|\psi|^2$

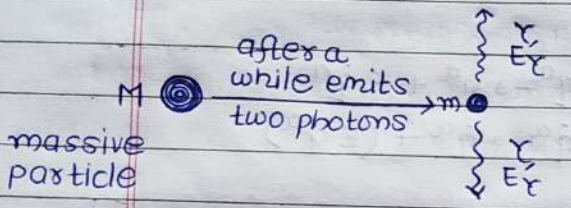
- M.E.'s → $F_{\mu\nu}$, J_ν are invariant under a certain symmetry & the symmetry is $x'^\mu = \Lambda^\mu_\nu x^\nu$

Then we constructed momentum $p^\mu = \left(\frac{E}{c}, p^x, 0, 0\right)$ for a wave travelling along x-axis



- Arguments for $E = mc^2$

① Einstein (1905, improved 1947)



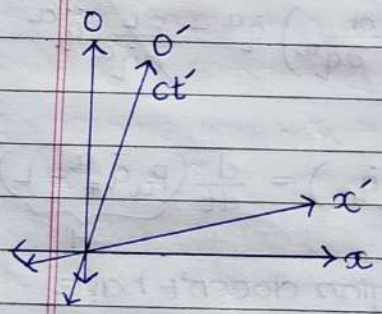
massive particle

$E'_M = E'_m + 2E'_\gamma$ ← energy of photon

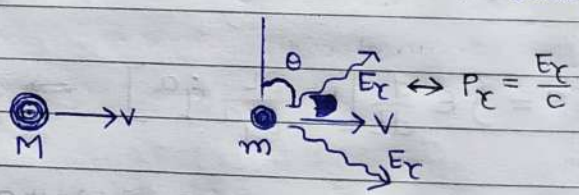
$p'^\mu = \left(\frac{E'_\gamma}{c}, 0, 0, \frac{E'_\gamma}{c}\right)$

i) we work in a frame where M & m are at rest, say O'

ii) we transform to a frame where M is moving with velocity v, frame O



Lets assume that momentum is conserved in O



$ct = \gamma(ct' + \frac{v}{c}x')$

$p^\mu = \left(\gamma\frac{E'_\gamma}{c}, \dots\right)$ $\because x' = 0$

$\Rightarrow E_\gamma = E'_\gamma \gamma$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$ $\Rightarrow E_\gamma \approx E'_\gamma$ to order v^2

↑ $O\left(\left(\frac{v}{c}\right)^2\right)$

(x-component) $Mv = mv + 2\frac{E_\gamma}{c}\cos\theta \sin\theta$ But $\sin\theta \approx \frac{v}{c}$ if $v \ll c$

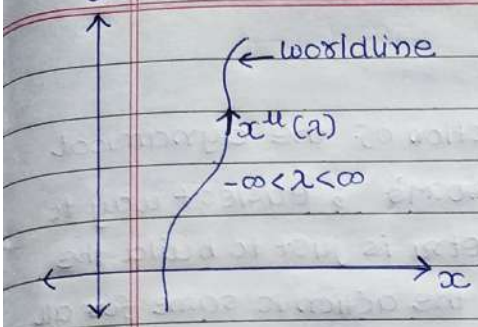
$Mv = mv + 2\frac{E_\gamma}{c}\cdot\frac{v}{c}$ for $v \ll c$

$\Rightarrow (M-m)c^2 = 2E_\gamma$ $\Rightarrow$ $\Delta M c^2$ = energy released in photons

in particular, if $m=0$ then E released $= Mc^2$

Particles and their action :

Lorentz invariant properties :



we can ask at any particular point on worldline, what's the velocity of that worldline ?

we can form a quantity which is parameter velocity of worldline.

$$\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \eta_{\mu\nu}\frac{dx'^\mu}{d\lambda}\frac{dx'^\nu}{d\lambda} \quad \text{under L.T.'s}$$

<0	timeline
$=0$	null $\lvert dx\rvert = cdt$
>0	spacelike (tachyon)

- Massive particles follow timelike trajectories ; we can define "proper time" along worldline $d\tau = \sqrt{dt^2 - \frac{dx^2}{c^2}} = dt\sqrt{1-\left(\frac{dx^2}{dt}\right)\frac{1}{c^2}}$

$$d\tau = dt\sqrt{1-v^2/c^2} \quad \text{--- lorentz invariant}$$

we can define $U^\mu = \frac{dx^\mu}{d\tau}$ (four-velocity)

Lorentz vector / Lorentz vector / Lorentz invariant

$$\eta_{00}\frac{U^0U^0}{c^2} = -\frac{d(ct)^2}{d\tau^2} + \left(\frac{dx}{d\tau}\right)^2$$

$$= -\frac{d(t)^2}{d\tau^2} + \frac{(dx/d\tau)^2}{c^2} = -1$$

putting value of $d\tau$

$$\eta_{\mu\nu}\frac{U^\mu U^\nu}{c^2} = -1$$

$m_0U^\mu = p^\mu$ (4-momentum) ; so this is also Lorentz 4-vector

m_0 : rest mass (same in every frame)

$$\Rightarrow \eta_{\mu\nu}p^\mu p^\nu = -m_0^2 = -\left(\frac{E}{c}\right)^2 + p^2$$

$$p^\mu = \left(\frac{E}{c}, p\right) \Rightarrow E = p^0c = \sqrt{p^2c^2 + m_0^2c^4}$$

If $p^2c^2 \ll m_0^2c^4 \longrightarrow E \approx m_0c^2 + \frac{1}{2}p^2/m_0^2 + \cdots$ Expanding squareroot in Taylor series

Also $p = m_0U = m_0\frac{dx}{dt}\cdot\frac{dt}{d\tau} = \gamma m_0\frac{dx}{dt} = \gamma m_0 v$

Remember : $v = \frac{dx}{dt}$; $U = \frac{dx}{d\tau}$

for $v \ll c$, $p \approx m_0v$

$\therefore E \approx m_0c^2 + \frac{1}{2}m_0v^2 + \cdots$ m_0c^2 is just a constant here, which we can ignore; this doesn't prove $E = mc^2$, but a clue

→ Action for a relativistic particle :

(i) Action is a variational principle, of some function of the dynamical trajectory, whose variation $\delta S = 0$ gives E.O.M's ; easiest way to ensure that these E.O.M's have some symmetry is just to build the symmetry into the action ie. to make sure that the action is same for all trajectories related by the symmetry

If we consider dx^μ to be this small 4-vector between two points, then $\eta_{\mu\nu} dx^\mu dx^\nu$ is the invariant quantity, provided that we build our action out of this quantity, then it'll be lorentz invariant

After parametrizing, we can write $\int_{\lambda_i}^{\lambda_f} d\lambda\, \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$

Is that expression good ? well, it has lorentz invariance (providing λ doesn't change while we do L.T.) ; but it has a nasty property, there should be no preferred λ (parametrizat^n of line shouldn't really matter), we can have any parameter, as long as it varies monotonically with the line, so $\lambda = \tilde{\lambda}$ (some other parameter)

In the expression, powers of $d\lambda$ don't match $d\lambda\, \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \neq d\tilde{\lambda}\, \frac{\eta_{\mu\nu} dx^\mu}{d\tilde{\lambda}} \frac{dx^\nu}{d\tilde{\lambda}}$

so it's not good when our action depends on how we parametrize the line, so we take squareroot to make powers equal

$$\int_{\lambda_i}^{\lambda_f} d\lambda \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} = \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \int d\tau \qquad \text{(action)}$$

for a timelike particle that quantity will be +ve & square root is well defined

$\therefore S = \int d\tau$ ← free particle action

natural --- action for a worldline

we can put any constant we want (out of the integral) & it turns out

that the natural constant to use is $-m_0 c$

- Let's see what action do we get for a particle that couples with electromagnetism ie. electrically charge particle

How can we couple the worldline with electromagnetic fields?

well we can use $F_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$; But somethings wrong with it

($F_{\mu\nu}$: antisymmetric, $\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$: symmetric) ; we can use $A_\mu \frac{dx^\mu}{d\lambda}$

$$\therefore S = -m_0 c \int_{\lambda_i}^{\lambda_f} d\lambda \left[\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\right] + \int d\lambda\, q A_\mu(x^\mu(\lambda))\frac{dx^\mu}{d\lambda}$$

$A_\mu(x^\mu(\lambda))$: vector potential on worldline ie. at the spacetime point the particle is

$m\frac{d^2x}{dt^2} = q(E + v\times B)$ --- E.O.M of a particle in electromagnetic field

↑ we should get this after varying the action

Let's vary S : $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$

↑ small

(for simplicity, ignore $d\lambda$)

$$dS = \int \frac{m_0 c\, \eta_{\mu\nu} dx^\mu d\delta x^\mu}{\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}} = c d\tau$$

$$dS = \int d\tau\, m_0 \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\mu)}{d\tau} + q\,\underbrace{\partial_\nu A_\mu}_{\delta(A_\mu(x))}\delta x^\nu \frac{dx^\mu}{d\tau}d\tau + \underbrace{qA_\mu\frac{d}{d\tau}\delta x^\mu d\tau}$$

Integrating by parts $\quad -q\frac{d}{d\tau}A_\mu \delta x^\mu = \partial_\nu A_\mu \frac{dx^\mu}{d\tau}\delta x^\mu$

$$dS = -m_0\frac{d^2x^\nu}{d\tau^2}\delta x^\nu + q(\partial_\nu A_\mu - \partial_\mu A_\nu)\delta x^\nu \frac{dx^\mu}{d\tau} = 0$$

$$dS = \int d\tau\, \delta x^\nu \left(-m_0\frac{d^2x_\nu}{d\tau^2} + q(\partial_\nu A_\mu - \partial_\mu A_\nu)\frac{dx^\mu}{d\tau}\right) = 0 \quad \forall\, \delta x^\nu$$

$$\Rightarrow \boxed{m_0\frac{d^2x_\nu}{d\tau^2} = qF_{\nu\mu}\frac{dx^\mu}{d\tau}} \text{ --- Lorentz Force law}$$

for calculations watch the lecture again ; no. 4

→ Action for electromagnetism:

$$S = \int d^4x \left(-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu \right)$$

$$\delta S = \int \left(-\frac{1}{2\mu_0} F^{\mu\nu} \underbrace{\delta F_{\mu\nu}}_{\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu} + J^\mu \delta A_\mu \right)$$

integrating by parts

$$\frac{1}{\mu_0} (\partial_\mu F^{\mu\nu}) \delta A_\nu + J^\nu \delta A_\nu \overset{!}{=} 0 \quad \forall\ \delta A_\nu$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu} \quad \text{--- Maxwell eqn}$$

we have reduced Maxwell's eqn in a neater form using $\eta_{\mu\nu}$; $\eta_{\mu\nu}$ was needed because some quantities in M.E's had lower indices & some upper indices & we use η to raise & lower indices and that allows us to introduce minus sign & then we say that all the laws of physics will be preserved in a transformatn where η is invariant

- Rotations leave the lengths invariant & lengths are sum of squares of the components (Pythagoras rule), which implicitly uses δ_{ij} in cartesian coordinates) $x^2 = x_i x_j \delta_{ij}$

so when we go to space-time & we have a minus sign, we write

$$x^2 = \eta_{\mu\nu} x^\mu x^\nu$$

so η is anologue of δ in spacetime

Poincare group : group of coordinate transformations (twice differentiable) which leave $\eta_{\mu\nu}$ unchanged

coordinate transformatn $x^\mu \to x'^\mu(x^\alpha)$ can be differentiated twice

$$F'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$$

requires $\eta'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \eta^{\alpha\beta} = \eta^{\mu\nu} \Rightarrow$ Lorentz transformations

Lets consider $x^\mu \to x'^\mu(x^\nu)$

satisfying $\eta_{\gamma\delta} = \eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta}$

$\because dx'^\mu = \frac{\partial x'^\mu}{\partial x^\beta} dx^\beta \quad ; \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial}{\partial x^\mu}$

Claim: most general such coordinate transformation is an element of Poincare group $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ with $\Lambda^\mu{}_\nu \Lambda^\gamma{}_\alpha \eta_{\mu\gamma} = \eta_{\nu\alpha}$ or $\Lambda^\mu{}_\nu \Lambda^\alpha{}_\beta \eta^{\nu\alpha\beta} = \eta^{\mu\alpha}$

Proof: $\frac{\partial}{\partial x^\epsilon} \eta_{\gamma\delta} = 0 \Rightarrow \eta_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\epsilon} \frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\epsilon} = 0$

Add with $\gamma \leftrightarrow \epsilon$ since that term is symmetrical under $\gamma \leftrightarrow \epsilon$, we can add & substract with $\delta \leftrightarrow \epsilon$ (Reimann's trick.)

$$\Rightarrow \eta_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\epsilon} \frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\epsilon \partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\epsilon} \frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\gamma} +$$

$$\eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\epsilon} - \eta_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\delta} \frac{\partial x'^\beta}{\partial x^\epsilon} - \eta_{\alpha\beta} \frac{\partial x'^\gamma}{\partial x^\gamma} \frac{\partial^2 x'^\beta}{\partial x^\epsilon \partial x^\delta}$$

↑ Relabel indices $\alpha \leftrightarrow \beta$ we get

$-\frac{\eta_{\beta\alpha}}{\eta_{\alpha\beta}} \frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\delta} \frac{\partial x'^\alpha}{\partial x^\epsilon}$ (∵ invariant) & term gets cancelled

$$\Rightarrow \eta_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\epsilon} \frac{\partial x'^\beta}{\partial x^\delta} = 0$$

To get rid of the Jacobian term $\frac{\partial x'^\beta}{\partial x^\delta}$, we multiply by its inverse $\frac{\partial x^\delta}{\partial x'^\mu}$,

use $\frac{\partial x'^\beta}{\partial x^\delta} \frac{\partial x^\delta}{\partial x'^\mu} = \frac{\partial x'^\beta}{\partial x'^\mu} = \delta^\beta_\mu$

$$\Rightarrow \eta_{\alpha\mu} \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\epsilon} \Rightarrow \boxed{\frac{\partial^2 x'^\nu}{dx^\gamma dx^\epsilon} = 0}$$

↑ multiply by inverse of η ie. $\eta^{\mu\nu}$; $\eta_{\alpha\mu}\eta^{\mu\nu} = \delta^\nu{}_\alpha$

↑ satisfying this eqn.

so only coordinate transformat^n leaving η invariant are the ones

if the derivative of something is zero, then something has to be constant

$$\Rightarrow \frac{\partial x'^{\nu}}{\partial x^{\epsilon}} = \Lambda^{\nu}_{\ \epsilon} \quad \text{constant}$$

$$\rightarrow \boxed{x'^{\nu}(x) = \Lambda^{\nu}_{\ \epsilon} x^{\epsilon} + a^{\nu}} \quad \text{------Poincaré}$$

ie. most general coordinate system which would leave η invariant & therefore leave M.E.'s invariant is the Poincare group

- we see that η & Poincaré group are strongly tied together & Poincaré, group of translations & Lorentz translation transformation is the grp of symmetries of M.E.'s & that's the foundation of SR.

- In GR, we make things more general. Lets try to get rid of η (too special in M.E.'s, it's a funny constant matrix; why should it be fundamental), Let's write $\eta_{\mu\nu}$ to be something more general $g_{\mu\nu}(x)$: symmetric 4x4 matrix

- In SR, we use η to calculate invariant quantities ie. we draw a particle trajectory & ask what is the Lorentz invariant distance along the particle's trajectory

$\sqrt{-\eta_{\mu\nu} dx^{\mu} dx^{\nu}}$ is invariant ; so we are going to use $g_{\mu\nu}(x)$ to calculate invariant quantities

$\sqrt{-g_{\mu\nu} dx^{\mu} dx^{\nu}}$ & this will turn out to be way to introduce gravity into motion of a particle

$g_{\mu\nu}(x)$ describes the effect of gravity & tells us how to define (geometry) coordinate invariant length & angles and..... (curved geometry)

How should $g_{\mu\nu}$ transform under coordinate transformations ?

Idea is : $g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x)$

for $x^{\mu} \rightarrow x'^{\mu}(x^{\alpha})$ are arbitrary 1-1 mapping that (every x corresponds to a x' & vice versa) requires $\frac{\partial x'^{\alpha}}{\partial x^{\beta}}$ (Jacobian matrices) are invertible so that

$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} = \delta^{\alpha}_{\ \mu}$ etc.

Now, action for a relativistic particle becomes,

$$-mc\int\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

- Einstein's insight : if the action determining the motion of particles is just the length of their worldline then all we need to determine how particles fall in space & time, we need to know the length of worldline & if we know the metric that will determines motion of particles & that metric would determine gravity

- Gravity is encoded in $g_{\mu\nu}(x)$ & immediately we get the idea that everything's goanna fall same under gravity because everybody object sees the same $g_{\mu\nu}(x)$; its universal ie. object of any mass or composition fall in same way ; $g_{\mu\nu}(x)$ is the metric created by earth

In fact, uniform gravitational field may be ~~created or~~ removed by a coordinate transformation.

(If you sit on a roundabout and hold on to it, it feels that theirs gravity pushing you outwards, but thats just an artifact of the fact that you are sitting in a rotating coordinate system, staying at the same radius ; if somebody could describe you in a non-rotating coordinate system then they would use Newton's laws & there would be no funny fictitious forces ; other point of view is you go to rotating coordinate system & you see a fictitious force ; so these fictitious force is created by going to diff. coordinate system & that's what happens with gravity)

Let take a free particle in Minkowski spacetime

$$\frac{d^2x^\mu}{dt^2}=0 \quad \text{ie. particle following straight line parametrized by } \tau$$

$$\Longrightarrow x^\mu=(\tau,\tau v,0,0)\tau$$

timelike $\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=-1 \Longrightarrow \tau=\frac{1}{\sqrt{1-v^2}}$

From now, I'll use 'X' instead 'x'

Let's take this particle to new coordinates

$$x'^{\mu}(x^{\alpha}) \longleftrightarrow x^{\alpha}(x'^{\mu})$$

$$\frac{d^2x^{\alpha}}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{dx'^{\nu}}{d\tau}\right) = \frac{\partial^2 x^{\alpha}}{\partial x'^{\beta}\partial x'^{\nu}}\frac{dx'^{\beta}}{d\tau}\frac{dx'^{\nu}}{d\tau} + \frac{\partial}{\partial x'^{\beta}}\left(\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\right)\frac{dx'^{\beta}}{d\tau}$$

$$\frac{\partial x'^{\alpha}}{\partial x'^{\nu}}\frac{d^2x'^{\nu}}{d\tau^2} = 0$$

$$\Rightarrow \frac{d^2x'^{\nu}}{d\tau^2} + \underbrace{\frac{\partial x'^{\nu}}{\partial x^{\mu}}\frac{d^2x^{\mu}}{dx'^{\alpha}dx'^{\beta}}\frac{dx'^{\alpha}}{d\tau}\frac{dx'^{\beta}}{d\tau}}_{\text{fictitious force}} = 0$$ ---- so in the EOM of a particle we've created a force

$a = \text{const}$

Take $t' = t$; $x' = x + \frac{1}{2}at^2 \Rightarrow t' = \tau$

$x^{\mu} = (t, x)$; $c = 1$ $\qquad x' = \tau v\tau + \frac{1}{2}a(\tau)^2$ ← uniform acceleration

$x^{\mu} = (\tau, \tau v, 0, 0)\tau$ $\qquad y' = 0$

$z' = 0$

ie. trajectory of particle in x coordinates was a straight line which becomes a curved line in x' coordinates ; so the particle in the primed coordinate is undergoing acceleration ; just by a coordinate transformation we've made the particle to accelerate

$$-1 = \eta_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = \underbrace{\eta_{\mu\nu}\frac{dx^{\mu}}{dx'^{\alpha}}\frac{dx^{\nu}}{dx'^{\beta}}}_{g'_{\alpha\beta}(x')}\frac{dx'^{\alpha}}{d\tau}\frac{dx'^{\beta}}{d\tau}$$

$$g'_{\alpha\beta}(x') = \begin{pmatrix} -1+(at')^2 & -at' & & \\ -at' & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

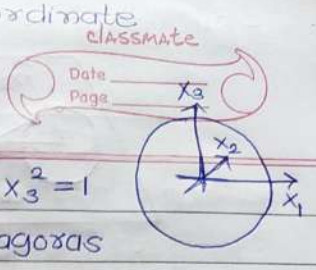
metric in the accelerated system of coordinates, responsible for fictitious force ; represents uniform gravitational field

- we did completely trivial, just changed the perspective and we suddenly get gravity ; so gravity can be created by going to an accelerated system

$t = t'$
$x = x' - \frac{1}{2}at'^2$

for 00 component : $-\frac{dx^0}{dx'^0}\frac{dx^0}{dx'^0} + \frac{dx^1}{dx'^0}\frac{dx^1}{dx'^0} = -1 + (at')^2$

Note: Unlike $\eta_{\alpha\beta}$, $g'_{\alpha\beta}(x')$ is not invariant under coordinate transformations

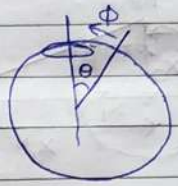


Example: consider pure space, no time

sphere S^2 i) embedded in $\mathbb{R}^3$ $\quad x^2 = x_1^2 + x_2^2 + x_3^2 = 1$

length given by $d\vec{x}^2$ in embedded space: Pythagoras

- The point of describing geometry of space using a metric is to describe it intrinsically (If you are an ant living on surface of the sphere, it really doesn't matter what's outside it or inside it, all you care about is what within the sphere) Can we describe sphere intrinsincally, by the quantities intrinsic to the sphere?



ii) intrinsically $x_1 = \sin\theta\cos\phi$

$x_2 = \sin\theta\sin\phi$

$x_3 = \cos\theta$

Now, how to define metric (distance in sphere)?

$$dx_1^2 + dx_2^2 + dx_3^2 = d\theta^2 + \sin^2\theta\, d\phi^2 = g_{ij}\, dx^i dx^j$$

where $x^i = (\theta, \phi)$, $i = 1, 2$ — metric on sphere

or write $g_{ij}\, dY^i dY^j$; $Y^i = (\theta, \phi)$ for intrinsic coordinates

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$ ---- induced metric on the sphere from the embedded metric

Are θ & ϕ are good coordinates on S^2?

Coordinates are map from space to coordinates, so there should be one-to-one correspondence ie., every point on space should have only one coordinate value

θ = const. — N $\theta = 0, \phi$ anything $\quad 0 \le \theta \le \pi$

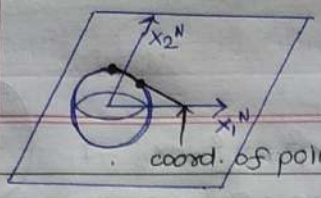
$0 \le \phi \le 2\pi$ with $0, 2\pi$ identified

At poles, we have one-to-many correspondence; bad coordinates

map $S^2 \to \{\theta, \phi\}$ is 1-to-many at N.P. & S.P

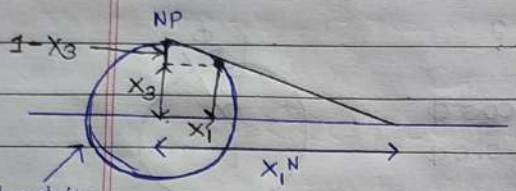
- By some topological reason, it's actually impossible to write a coordinate system for sphere that has one-to-one correspondence. So what we do is to use diff. coordinate systems for diff. parts of space.

- But there's better system of coordinates; projective coordinates (x_1^N, x_2^N)



singularity; for N.P., the coord. will be at ∞ & for a point in neighbourhood of N.P., coordinate will be at large radius

eg. consider a point on great circle $\phi = 0$ $x_2^N = 0$



similar triangles: $\frac{x_1^N}{1} = \frac{x_1}{1-x_3}$

$\Rightarrow x_1^N = \frac{x_1}{1-x_3}$

so at NP $x_3 = 1$; $x_3 \to 1$ $x_1^N \to \infty$

Generalize to arbitrary x_1, x_2 : $x_1^N = \frac{x_1}{1-x_3}$ $\quad x_2^N = \frac{x_2}{1-x_3}$ projective coordinates in terms of ~~embedding coord.~~

Likewise, we have SP projective coordinates : $x_1^S = \frac{x_1}{1+x_3}$; $x_2^S = \frac{x_2}{1+x_3}$

where $x_1^N = \frac{x_1^S}{(x_1^S)^2 + (x_2^S)^2}$, $x_2^N = \frac{+x_2^S}{(x_1^S)^2 + (x_2^S)^2}$

metric $dx^2 = \frac{4\left[(dx_1^N)^2 + (dx_2^N)^2\right]}{\left[1 + (x_1^N)^2 + (x_2^N)^2\right]^2} = \frac{4\left[(dx_1^S)^2 + (dx_2^S)^2\right]}{\left[1 + (x_1^S)^2 + (x_2^S)^2\right]^2}$

In general, we need a no. of coordinate systems to fully describe a curved space & the number fully depends on the topology.

Given some $g_{\mu\nu}(x)$, can I change coordinates to make $g_{\mu\nu}(x) \approx \eta_{\mu\nu}$ in my neighbourhood?

→ Yes

Assume $g_{\mu\nu}(x)$ is differentiable (we call a curved space a manifold & if it has a metric that is differentiable, we call it differentiable manifold)

∴ $g_{\mu\nu}(x)$ will have a Taylor expansion. choose coordinates in which I am located at $x=0$

$$g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}g_{\mu\nu,\lambda\sigma}(0)x^\lambda x^\sigma + \cdots \quad \text{—— (1)}$$

↑ Taylor expansion in D = no. of ind. coordinates so that $x^\mu = (x^0, x^1, \ldots, x^{D-1})$

can I choose a function $x'^\alpha(x)$ s.t. $g'_{\mu\nu}(0) = \eta_{\mu\nu}$, $g'_{\mu\nu,\lambda}(0) = 0$?

→ Yes, we can always find coordinates in which metric is $\eta_{\mu\nu}$ & the first derivative is zero

$x'^\alpha(x) \to x^\alpha(x'^\mu)$ if x' is function of x, then x is also funct^n of x'; one-to-one map

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\frac{dx^\alpha}{dx'^\mu}\frac{dx^\beta}{dx'^\nu} \quad \text{—— transformat}^n \text{ law going from } x \to x' \quad \text{—— (3)}$$

→ Taylor expansion of x' around '0' with the assumption that the point $x'=0$ goes to $x=0$; assume coordinate transformat^n itself is a Taylor series

$$x^\alpha(x') = k^\alpha_\tau x'^\tau + \frac{1}{2}L^\alpha_{\tau\sigma}x'^\tau x'^\sigma + \frac{1}{6}M^\alpha_{\tau\sigma\lambda}x'^\tau x'^\sigma x'^\lambda + \cdots \quad \text{—— (2)}$$

↑ arbitrary const. matrix

$$\left(k^\alpha_\tau = \frac{\partial x^\alpha}{\partial x'^\tau}\right)_{x'=0} ; \left(L^\alpha_{\tau\sigma} = \frac{\partial^2 x^\alpha}{\partial x'^\tau \partial x'^\sigma}\right)_{x'=0}$$

Using (1) & (2), this is a Taylor series in x'^μ

First term in Taylor series ind. of x':

$$g'_{\mu\nu}(0) = g_{\text{row}\,\alpha\beta\,\text{column}}(0)\,k^{\text{row}\ \alpha}_\mu\,k^{\text{column}\ \beta}_\nu \overset{?}{=} \eta_{\mu\nu} \qquad (k^T)^\mu_\alpha = k^\alpha_\mu$$

In matrix form, $k^T g(0) k = \eta$ ---- want to solve this

Let $k = O\Lambda$ ← diagonal matrix

↑ orthogonal matrix

$\exists\ O$ s.t. $O^T g O = \text{diagonal}$

$$O^T g O = \text{diagonal}\begin{pmatrix} -\lambda_0^2 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & \lambda_2^2 & 0 \\ 0 & 0 & 0 & \lambda_3^2 \end{pmatrix}$$

∵ g is symmetric, there always exist an orthogonal matrix that makes it diagonal

"psuedo - Reimannian"

Fundamental ass. of GR : $g_{\mu\nu}(x)$ is a symmetric matrix with one eigenvalue negative & three positive eigenvalues

ie. locally space is gonna be flat & in that flat space one direction is gonna be timelike & three are spacelike

(Reimann in Reimannian geometry made the assumption that all of the eigenvalues are +ve, so he was interested in spaces that are locally Euclidean ; Einstein want spacetime to look locally like Minkowski space, having one -ve & 3 +ve eigenvalues)

$$k^T g k = L O^T g O L = L \begin{pmatrix} -\lambda_0^2 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & \lambda_2^2 & \\ 0 & 0 & 0 & \lambda_3^2 \end{pmatrix} L$$

$$k^T g k = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{if we choose } L = \begin{pmatrix} 1/\lambda_0^2 & 0 & 0 & 0 \\ 0 & 1/\lambda_1^2 & 0 & 0 \\ 0 & 0 & 1/\lambda_2^2 & 0 \\ 0 & 0 & 0 & 1/\lambda_3^2 \end{pmatrix}$$

$\Rightarrow$ There are enough parameters in k^μ_ν to set an arbitrary $g_{\mu\nu}$ to $\eta_{\mu\nu}$ at the point $x = x' = 0$

- $g_{\alpha\beta}$ has 10 parameters ; k has 16 parameters (doesn't have any symmetry) $\Rightarrow$ we have more parameter than we need, so we can easily set $g_{\mu\nu} = \eta_{\mu\nu}$; other 6 parameters in k are Lorentz transforms (3 rotations + 3 boosts)

Can I set $g'_{\mu\nu,\lambda}(0) = 0$ ie. $\frac{\partial}{\partial x'^\lambda} g'_{\mu\nu}(x') = 0$ using parameters in $L_{\gamma\delta}$

$$g'_{\mu\nu,\lambda}(0) = g_{\alpha\beta,\delta}(0) k^\delta_\lambda k^\gamma_\mu k^\beta_\nu + g_{\alpha\beta}(0)\left(L^\alpha_{\mu\lambda} k^\beta_\nu + k^\gamma_\beta L^\beta_{\mu\nu}\right)$$

can I solve $g'_{\mu\nu,\lambda}(0) = 0$ for $L^\gamma_{\mu\lambda}$?

$g'_{\mu\nu,\lambda}$ $\leftarrow D$; symmetric $\frac{D(D+1)}{2}$ comp.

$L^{\alpha}_{\mu\lambda}$ $\leftarrow D$; symmetric $\rightarrow D \cdot \frac{D(D+1)}{2}$ parameters

$D \cdot \frac{D(D+1)}{2}$ equations

(Reimann's trick from last time is exactly manipulat^n needed to solve explicitly for L)

Now let's calculate $g'_{\mu\nu,\alpha\beta}(0)$ = eqn linear in M

$g'_{\mu\nu,\alpha\beta}$ = $\frac{D^2(D+1)^2}{4}$ equations

$\frac{D(D+1)}{2}$ p.d. commute → symmetric $\frac{D(D+1)}{2}$

$M^{\alpha}_{\beta\delta\lambda}$ ← D

p.d. commute symmetric $\frac{D(D+1)(D+2)}{6} = \frac{D^2(D+1)(D+2)}{6}$ parameters ; so we do not have enough parameters

∴ no. of components of $g'_{\mu\nu,\alpha\beta}$ that cannot be cancelled by $M^{\alpha}_{\beta\delta\lambda}$

$= \frac{D^2(D+1)^2}{4} - \frac{D^2(D+1)(D+2)}{6} = \frac{D^2(D^2-1)}{12}$ = no. of components of a new tensor, Reimann tensor, measuring the curvature of spacetime

for $D=1 \to 0$; 1D geometry is just a curve, curve can't have any intrinsic curvature, there's no way of knowing whether curve is a curve within the space

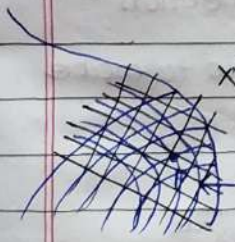
$D=2 \to 1$; 1 2nd derivative of the metric that cannot be set = 0 ; there's one quantity measuring the curvature of space

$D=3 \to 6$.

$D=4 \to 20$; Reimann Tensor will have 20 quantities whose 2nd derivative we cannot set = 0 under a coordinate transformation

Tensors under general coordinate transformations ;

eg. i) scalar $\phi'(x') \to \phi(x)$ ie. value of scalar at a given point is equal to its value at same point in old coordinate system



ii) vectors $V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}(x)$; $V'_{\mu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha}(x)$

no. of upstairs (contravariant) indices & no. of downstairs (covariant) indices determine the type of tensor

eg. $T^{\mu\nu}{}_{\lambda\beta\sigma}$ = tensor of type (2,3) ← how many Jacobian matrices & inverse Jacobian matrices you need to describe the transformation laws

- metric $g_{\mu\nu}$ is a tensor of type (0,2)

- Tensors from tensors:

Add: $T_{\mu\nu} + S_{\mu\nu}$ is a tensor $M_{\mu\nu}$ of same type

Multiply: $T^{\mu\nu} S_{\alpha\beta}$ is a tensor $M^{\mu\nu}{}_{\alpha\beta}$ of type (2,2)

Contract: $T^{\mu\nu} S_{\nu\alpha}$ is a tensor $M^{\mu}{}_{\alpha}$ of type (1,1)

Differentiate: $\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}}$ type (0,1)

$\rightarrow V_{\mu} = \frac{\partial \phi(x)}{\partial x^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha}$

$V'_{\mu} = \frac{\partial \phi'(x')}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}} \phi(x)$ tensor of type (0,1)

$\rightarrow B_{\mu\nu} = \partial_{\mu} A_{\nu}$

$B'_{\mu\nu} = \partial'_{\mu} A'_{\nu} = \partial'_{\mu} \left(\frac{\partial x^{\alpha}}{\partial x'^{\nu}} A_{\alpha} \right) = \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \partial_{\beta} A_{\alpha} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} A_{\alpha}$ BAD

That's nothing like a tensor transformatn law

$B'_{\mu\nu}$ is not a tensor

Need a fix: Redefine the derivative as follows

$\nabla_{\mu} A_{\nu} = \partial_{\mu} A_{\nu} - \Gamma^{\lambda}_{\nu\mu} A_{\lambda}$ extra piece Γ is the "connection"

↑ "covariant derivative"

↑ its not a tensor; its role is to make derivative into a tensor

Transformatn for Γ: $\Gamma'^{\nu}_{\alpha\mu} = \frac{\partial x'^{\nu}}{\partial x^{\delta}} \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \Gamma^{\delta}_{\lambda\sigma} + \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial^2 x^{\beta}}{\partial x'^{\alpha} \partial x'^{\mu}}$

$$\partial_\mu A_\nu - \Gamma^\lambda_{\nu\mu} A_\lambda$$

$$\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} A_\alpha - \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\lambda} A_\beta = \frac{\partial x'^\lambda}{dx^\alpha} \frac{\partial x^\beta}{\partial x'^\lambda} = \delta^\beta_\alpha$$

$$- \frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\nu} A_\beta$$

$$\Rightarrow (\nabla_\mu A_\nu)' = \nabla'_\mu A'_\nu = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \nabla_\alpha A_\beta \qquad \text{tensor of type } (0,2)$$

What about covariant derivative of $\nabla_\mu V^\nu$?

Again $\partial_\mu V^\nu$ is not a tensor so we introduce a Γ correction

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\lambda\mu} V^\lambda \quad ; \text{ turns out to be tensor of type } (1,1)$$

2nd term in $\Gamma'^\nu_{\alpha\mu}$ $\frac{\partial x'^\nu}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial x'^\alpha \partial x'^\mu}$, re-express in terms of $\frac{\partial^2 x'}{\partial x \partial x}$

$$\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\nu(x)}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'^\alpha}\right) = 0$$

$$\frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} + \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'^\alpha} = 0$$

$$\Rightarrow \frac{\partial x'^\nu}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial x'^\alpha \partial x'^\mu} = - \frac{\partial^2 x'^\nu}{\partial x^\gamma \partial x^\beta} \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\beta}{\partial x'^\nu}$$

In general, if $A^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}$ is a tensor

then $\nabla_\mu A^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} = \partial_\mu A^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} + \Gamma^\alpha_{\lambda\mu} A^{\lambda\beta\ldots}{}_{\gamma\delta\ldots} + \Gamma^\beta_{\lambda\mu} A^{\alpha\lambda\ldots}{}_{\gamma\delta\ldots} + \ldots$

$$- \Gamma^\lambda_{\mu\gamma} A^{\alpha\beta\ldots}{}_{\lambda\delta\ldots} \quad \text{etc}$$

what is Γ ?

i) $\Gamma^\lambda_{[\alpha\beta]} = \frac{1}{2}\left(\Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{\beta\alpha}\right)$ ---- antisymmetric part of Γ is a tensor (Bad term in the transformat^n is symmetric under $\alpha \leftrightarrow \beta$) of Γ

we call this the torsion $T^\lambda_{\alpha\beta}$; simplest assumption we could make would be $T^\lambda_{\alpha\beta} = 0$

Obvious but powerful theorem ; If a tensor is zero in one coordinate system, then its zero in all coord. systems

- If we assume $T^{\mu}_{\alpha\beta} = 0$, then $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$ & Γ can be determined from a simple condition ; $\nabla_\mu g_{\alpha\beta} = 0$

"METRICITY"

I want write down an eq^n that connects Γ with 1st derivatives of metric because 1st derivatives were flaky things we could remove by coordinate trans^n & Γ is a flaky thing & we want to connect these two things

ie. the quantity we use to fix the geometry, the metric, doesn't itself have a first derivative

ie. if you take the dot product between two vectors at one point & if those two vectors are constants, the dot product is not goanna change

$$g_{\alpha\beta} U^\alpha V^\beta = U.V$$

if $\nabla_\mu U^\alpha = 0$ & $\nabla_\mu V^\beta = 0$

$\nabla_\mu (U.V) = \partial_\mu (U.V)$ covariant derivative obeys the same law as an ordinary derivative ; distributive

$$\nabla_\mu (g_{\alpha\beta} U^\alpha V^\beta) = (\nabla_\mu g_{\alpha\beta}) U^\alpha V^\beta + g_{\alpha\beta} (\nabla_\mu U^\alpha) V^\beta + g_{\alpha\beta} U^\alpha (\nabla_\mu V^\beta)$$

$\Rightarrow \nabla_\mu g_{\alpha\beta} = 0$ ---- metric doesn't change covariantly across spacetime

- so the claim is if we apply metricity condition on the metric, it fixes Γ

$$\partial_\mu g_{\alpha\beta} - \Gamma^{\lambda}_{\alpha\mu} g_{\lambda\beta} - \Gamma^{\lambda}_{\beta\mu} g_{\lambda\alpha} = ?$$

Γ is symmetrical under change of α & β ; looks like L , so to solve for Γ add two more eq^n, we get

$$\partial_\mu g_{\alpha\beta} - \Gamma^{\lambda}_{\alpha\mu} g_{\lambda\beta} - \Gamma^{\lambda}_{\beta\mu} g_{\lambda\alpha} + \partial g - \Gamma g - \Gamma g - \partial g + \Gamma g + \Gamma g = 0$$

$$\Gamma^{\lambda}_{\alpha\mu} = \frac{1}{2} g^{\lambda\beta} \left(g_{\alpha\beta,\mu} + g_{\mu\beta,\alpha} - g_{\alpha\mu,\beta} \right) = \left\{ {\lambda \atop \alpha\mu} \right\}$$ Christoffel symbol

$\Rightarrow$ In a coordinate system where 1st derivative of the metric is zero, Γ is going to be zero ; but its not a tensor so there's no reason for Γ to be zero in other coord. systems

- Remember that in a uniformly accelerating coordinate system, a freely moving particle actually accelerates & that acceleration comes from Γ ; Γ is the fictitious gravitational force created by a coordinate transformation ; it can also be removed by a coord. transformat^n, it takes you back to freely falling system

- So locally in any region of spacetime, gravity doesn't really exists ; Everything is SR, gravity can be removed ; The only way gravity comes into physics if we look at regions of space big enough that the curvature of spacetime matters, 2nd derivatives can't be removed & becomes significant & that's all described by the Reimann tensor

- so the connection in Maxwell's theory was 'A_μ' which is analogous to Γ ; physical quantity $F_{\mu\nu}$ is analogous to Reimann tensor ; the way $F_{\mu\nu}$ was written in derivative of 'A', Riemann tensor is written in terms of an antisymmetrized derivative of Γ

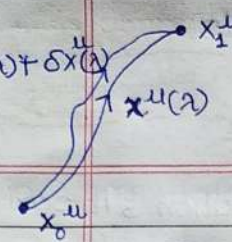
Geodesics : shortest path along a curved space

Action for a particle in Minkowski spacetime = $-m_0 \int \sqrt{-h_{\mu\nu}} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda$ ~~Lorentz-invariant~~ length of worldline $x^\mu(\lambda)$

in general spacetime = $-m_0 \int \sqrt{-g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$ --- proper length of worldline; ~~ind. of coord.~~

- so its amazing that the only effect of gravity is to change the geometry & then the trajectories of every particle moving in that geometry follow the same curve & the curve is obtained by just by extremizing the length of the worldline

Why does the classical world extremize something ? Best answer comes from QM ; If we formulate things in path integrals, we see that the phase associated with any trajectory is $e^{iS/\hbar}$ & if the S is an extremum then it means all the phases line up to add & get constructive interference



$$\delta S = -m_0 \int \frac{1}{2} \frac{d\lambda}{\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}} \Big(-\overbrace{g_{\mu\nu}}^{\delta g_{\mu\nu}}{}_{,\alpha}\,\delta x^\alpha \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\Big)$$

Re-express in terms of $d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$ & $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ etc

$$\delta S = m_0 \int d\tau \frac{1}{2}\Big(g_{\mu\nu,\alpha}\,\delta x^\alpha \dot{x}^\mu \dot{x}^\nu + 2\underbrace{g_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu}\Big) \quad \text{integrating by parts}$$

Relabel $\nu \to \lambda$; $\alpha \to \nu$

$$-2g_{\mu\nu}\ddot{x}^\mu \delta x^\nu - \underbrace{2g_{\mu\nu,\lambda}\dot{x}^\lambda \delta x^\nu \dot{x}^\mu} - (g_{\mu\nu,\lambda}+g_{\lambda\nu,\mu})\dot{x}^\mu\dot{x}^\lambda$$

Multiply by $g^{\nu\alpha}$ to get $\Gamma^\alpha_{\mu\lambda}$

$$\Rightarrow g_{\mu\nu}\ddot{x}^\mu + \frac{1}{2}\big(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\big)\dot{x}^\mu\dot{x}^\lambda = 0 \quad ; \forall\, \delta x^\nu$$

$$\ddot{x}^\mu + \underbrace{\frac{1}{2}g^{\alpha\nu}\big(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\big)}_{\text{Torsion-free } \Gamma^\alpha_{\mu\lambda}}\dot{x}^\mu\dot{x}^\lambda = 0$$

∴ classical path which is an extremum of the proper length satisfies

$$\boxed{\ddot{x}^\alpha + \Gamma^\alpha_{\mu\lambda}\dot{x}^\mu\dot{x}^\lambda = 0}$$ ----- Geodesic eqn

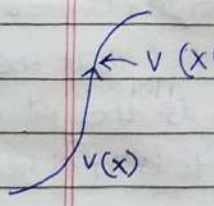
Guaranteed to be a tensor eqn because it followed from action principle respecting general coordinate invariance

$\dot{x}^\alpha$ is a tensor ; $\frac{dx'^\mu}{d\tau} = \frac{dx'^\mu}{dx^\nu}\frac{dx^\nu}{d\tau}$; for 2nd derivative it won't give a tensor

- Γ is just follows from varying an action which is invariant under coordinate transformations

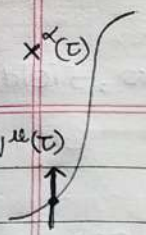
Here what we did is $\frac{d}{d\tau}V^\mu(x(\tau)) + \Gamma^\mu_{\alpha\beta}V^\alpha\frac{dx^\beta}{d\tau} = \frac{D}{D\tau}V^\mu(x(\tau))$

here consider ↑ ; ↑ "covariant derivative of V along curve $x(\tau)$"



This is not a tensor ; because V'^μ involves Jacobian matrix $\frac{dx'}{dx}$ & differentiating it will give a bad term

Useful to consider "parallel transport along a curve"



we say $V^\mu(\tau)$ is parallel transported along $x^\mu(\tau)$ if $\frac{DV^\mu}{D\tau} = 0$

Its guaranteed that if you do go to a locally flat system, with metric $\eta_{\mu\nu}$ at that particular point, Γ vanishes; so locally along every small neighbourhood along $V^\mu(\tau)$ we transport it a little bit ie. vector is actually constant in locally flat coordinates everywhere along the curve

- Likewise, for $U_\mu(\tau)$ $\frac{DU_\mu}{D\tau} = 0$, ie. $\frac{d}{d\tau} U_\mu(x(\tau)) - \Gamma^\lambda_{\mu\alpha} U^\mu \frac{dx^\alpha}{d\tau} = 0$

If $\frac{DV^\mu}{D\tau} = 0$ & $\frac{DU_\mu}{D\tau} = 0$, then $\frac{d}{d\tau}(V^\mu U_\mu) = 0 \quad \forall\, U, V \Rightarrow \Gamma^\lambda_{\mu\alpha} = \Gamma^\lambda_{\alpha\mu}$ ie. no torsion

$$\frac{D}{D\tau}(V^\mu U_\mu) = \underbrace{\frac{DV^\mu}{D\tau}}_{\left(\frac{dV}{d\tau} + \Gamma V\right)} U_\mu + V^\mu \underbrace{\frac{DU_\mu}{D\tau}}_{\left(\frac{dU}{d\tau} - \Gamma U\right)}$$

$\Rightarrow$ Requiring that parallel transported vectors have a constant inner product on any curve is the same as requiring that the torsion vanishes along that curve

- Riemann Tensor:

How do we build a tensor from the 2nd derivative of the metric?

We have c.d ∇_μ & we know how it acts on a quantity, say a vector $\nabla_\mu V^\mu$, but we want the result to be ind. of that quantity (in this case a vector); obviously 1st derivative of V^μ is not ind. of V^μ; so how can we build something out of derivitives acting on some quantity, so that the thing we built is ind. of that quantity?

Let's try to do electromagnetism

way to express c.d in electromagnetism is $\partial_\mu + iA_\mu$ → How do I get $\partial_\mu A_\nu - \partial_\nu A_\mu$ from $\partial_\mu + iA_\mu$

How do we get $F_{\mu\nu}$ from this? → we know $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

→ commute $[\partial_\mu + iA_\mu, \partial_\nu + iA_\nu]\phi$

$= i(\partial_\mu A_\nu - \partial_\nu A_\mu)$ $\quad [A_\mu, A_\nu] = 0$ because they are just numbers, fields

$= iF_{\mu\nu}\phi$

ie. commutator of derivatives gives field strength

so, lets take commutator of c.d.

$[\nabla_\mu, \nabla_\nu]V_\alpha \xrightarrow{\text{this will give}}$ tensor $\propto V_\alpha$

we can also act the commutator on a scalar, but it doesn't give any interesting results; so vector is the 1st obvious quantity to be operated upon

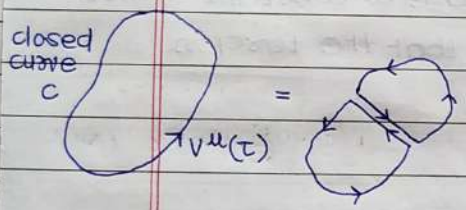
so $[\nabla_\mu, \nabla_\nu]V^\beta = +R^\beta_{\ \beta\mu\nu}V^\sigma$ (Riemann Tensor)

$[\nabla_\mu, \nabla_\nu]V_\beta = -R^\sigma_{\ \beta\mu\nu}V_\sigma$

Infact, if we act commutator on $V_{\beta\lambda\ldots}^{\ \ \delta\alpha\ldots}$ then we get +RVterm for each upstair index & -RVterm for each downstair index

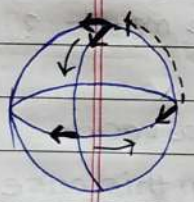
- so R allows us to compute commutator of c.d on any tensor

Riemann Tensor from Parallel transport; (Close geometrically)

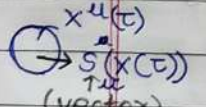


consider parallel transporting a vector around an infinitesimal closed curve

- If space is flat & we parallel transport a vector on it then its goanna come back to what it was, but when space is curved its not goanna do that



so what we are going to show that Riemann Tensor is exactly going to measure the fact that vectors donot come back to themselves when you take them around a closed loop.



← lets consider a small loop with curve $x^\mu(\tau)$ & vector $S_\mu(x(\tau))$ is parallel transported

$$\frac{dS_\mu}{d\tau} = \Gamma^\beta_{\mu\alpha}\frac{dx^\alpha}{d\tau}S_\beta$$

change in $S_\mu(\tau)$ as we go around the loop

$$\Delta S_\mu = \oint d\tau \frac{dS_\mu}{d\tau} = \oint \Gamma^\beta_{\mu\alpha}\frac{dx^\alpha}{d\tau}S_\beta\, d\tau = \oint \Gamma^\beta_{\mu\alpha}S_\beta\, dx^\alpha$$

for closed curves; $\oint dx^\alpha = 0$, geometrically →

Mathematically,

$$\oint d\tau \frac{dx^\alpha}{d\tau} = [x^\alpha]_{\tau_{in}}^{\tau_f} = 0$$

$\oint dx^\alpha dx^\sigma = a^{\alpha\sigma}$ ← its a tensor which depends on the curve

$= \int d\tau \frac{dx^\alpha}{d\tau} x^\sigma = -\int d\tau\, x^\alpha \frac{dx^\sigma}{d\tau}$ ... integrating by parts

proportional ~~to area of curve~~

∵ dimensions; L^2

$$\Rightarrow \oint dx^\alpha dx^\sigma = a^{\alpha\sigma} = -a^{\sigma\alpha} \leftarrow \text{"area element"}$$

infinitesimal curve, so we can Taylor series expansion for Γ around $x^\mu = 0$

centered on $x^\mu = 0$

$$\Gamma^\beta_{\mu\alpha} = \Gamma^\beta_{\mu\alpha}(0) + \partial_\lambda \Gamma^\beta_{\mu\alpha} x^\lambda + \cdots$$

$$S_\beta = S_\beta(0) + \partial_\nu S_\beta(0) x^\nu + \cdots$$

$$\Delta S_\mu = \oint \Gamma^\beta_{\mu\alpha} S_\beta dx^\alpha \approx \oint (\partial_\lambda \Gamma^\beta_{\mu\alpha} S_\beta + \Gamma^\beta_{\mu\alpha} \partial_\nu S_\beta) x^\mu dx^\mu$$

"area" 2nd order

can be replaced by $\Gamma^\kappa_{\beta\lambda} S_\alpha$ ← const. term vanishes $\oint dx = 0$

because S_α is parallel transported around the curve & dropping $x^\lambda x^\nu$ in integrand

$$\Delta S_\mu = \frac{1}{2} R^\beta_{\mu\lambda\alpha} S_\beta a^{\lambda\alpha}$$

↓ it's a tensor by construction, but one can check

where $\boxed{R^\beta_{\mu\lambda\alpha} = \partial_\lambda \Gamma^\beta_{\mu\alpha} + \Gamma^\delta_{\mu\alpha}\Gamma^\beta_{\delta\lambda} \quad (\lambda \leftrightarrow \alpha)}$

⇒ if I ~~to~~ follow some path (doesn't have to be geodesic, any path) in a manifold & I carry with me a vector & carry it around the path in a parallel manner, then the change when it comes back tells me something about the geometry of the manifold

Einstein's Field Equation for Gravity:

Recall the geodesic eq^n: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$; $\Gamma^\mu_{\nu\lambda} = \left\{ {\mu \atop \nu\lambda} \right\}$

↓ --- EOM ——— ①

ie. $\frac{DU^\mu}{D\tau} = 0 \equiv \frac{dU^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} U^\nu \frac{dx^\lambda}{d\tau}$ if U^μ is parallel transported along $x^\mu(\tau)$

- geodesic eq^n states that 4-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ is parallel transported along worldline.

for massive particles, we defined τ by $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$ ---(constrain ②

(test particle)

Eqn ① & ② gives the particles motion under influence of Gravity ; the idea of test particle is that its mass is not big enough to effect the gravity itself

$S = -m \cdot$ length of the worldline

Let's understand these eq^n go to Newtonian limit :

1) ie. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$ ... weak gravitational fields

2) Also assume that $|V| = \left|\frac{dx}{dt}\right|$ for particle is $\ll 1$ (in units where c

3) Assume that the stress-energy $T^{\mu\nu}$ creating the gravitational field has $|T^{ij}| \ll |T^{00}|$

pressure ↓

eg. for perfect fluid in its rest frame, $T^{ij} = P\delta^{ij}$

our assumption is $\left|\frac{P}{\rho_e}\right| \ll 1$

4) Assume $\frac{\partial}{\partial x^0}(g_{\mu\nu}) = 0$ if we fix the objects causing the fields, like sun, then the field around it is just const.

Taylor series for $(1+x) \approx 1-x$

Algebra: if $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ $\because (1+x)^{-1} = 1-x$

where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$

Let's check: $(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\beta} - \eta^{\nu\lambda}\eta^{\beta\delta}h_{\lambda\delta}) =$

$= \delta^\beta_\mu + h^\beta_\mu - h^\beta_\mu = O(h^2)$ .... zero to order h^2

$d\tau = \sqrt{dt^2 - dx^2} \approx dt\left(1 - \frac{1}{2}\left(\frac{dx}{dt}\right)^2\right) \approx dt$

so the 4-components of geodesic eq^n are

$$\frac{d^2x^i}{d\tau^2} \approx \frac{d^2x^i}{dt^2} \approx -\Gamma^i_{00}\left(\frac{dt}{d\tau}\right)^2 \approx -\Gamma^i_{00}$$

↑ acceleration ; ↑ F/m

$$\frac{d^2t}{d\tau^2} \approx -\Gamma^0_{00}\left(\frac{dt}{d\tau}\right)^2 \approx -\Gamma^0_{00}$$

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\alpha}\underbrace{\left(g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\lambda\nu,\alpha}\right)}_{O(h)} \approx \frac{1}{2}\eta^{\mu\alpha}\left(h_{\nu\alpha,\lambda} + h_{\lambda\alpha,\nu} - h_{\lambda\nu,\alpha}\right)$$

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; but $\eta_{\mu\nu}$ is a constant so derivatives will give zero ; so $g_{\mu\nu,\lambda}$ terms will be $O(h)$ & inverse also has $\eta^{\mu\nu}$ & $h^{\mu\nu}$ but can drop this 'h' term because we already have 'h' term due to $g_{\nu\alpha,\lambda}$; so if we kept it we would have 'h^2' and we're not interested in terms of $O(h^2)$.

$\therefore \Gamma^0_{00} \approx 0$ $\quad\because \frac{\partial}{\partial x^0} g_{\mu\nu} = 0$ $\quad \mu=0,\ \nu\lambda=00$ ie. $\eta^{\mu\alpha=0}_{\nu\lambda=00} \Rightarrow \alpha=0$

$\Gamma^i_{00} \approx -\frac{1}{2}\eta^{ij} h_{00,j} = -\frac{1}{2} h_{00,i}$ --- raising & lowering spatial indices doesn't involve any ~~minus sign~~

$= -\frac{1}{2}\frac{\partial}{\partial x^i} h_{00}$

$\Rightarrow \frac{d^2t}{d\tau^2} \approx 0$, consistent with $\tau \approx t$ ie. $dt \propto d\tau$, $\frac{dt}{d\tau} = \text{const.}$

from 1st eqn $\frac{d^2x^i}{dt^2} \approx -\Gamma^i_{00} \approx \frac{1}{2}\frac{\partial}{\partial x^i} h_{00}$

or $\frac{d^2x}{dt^2} \approx \frac{1}{2}\nabla h_{00}$

lets compare this with Newton, $m\frac{d^2x}{dt^2} = F_g = -m\nabla\phi_g$ ← gravitational potential

same mass, but unexplained ; so in Newton's theory $F \propto m$ for some ~~unknown reason~~

$\therefore$ In Newtonian limit $\left(|h_{\mu\nu}| << 1\ ,\ \left|\frac{v}{c}\right| << 1\right)$, $h_{00} = -2\phi_g$

$\Rightarrow$ Perturbation of the time-time component of the metric is Newton's gravitational potential

- so for mass M (like the earth or the sun)

$h_{00} \approx +\frac{2GM}{r}$ $\quad\therefore h_{00} \approx \frac{2GM}{rc^2}$

h_{00} = dimensionless ; $\frac{2GM}{r}$ = units of v^2

$g_{00} \approx -\left(1 - \frac{2GM}{rc^2}\right)$ something strange happens for $r \lesssim \frac{2GM}{c^2} \equiv$ Schwarzschild radius

Examples: i) surface of earth $|h_{00}| \approx 10^{-9}$

ii) ——"—— sun $\approx 10^{-6}$

iii) white dwarfs $\approx 10^{-4}$

iv) neutron star $\approx 10^{-1}$ v) Black hole ≈ 1

- All of the Newtonian gravity is geodesic eqn where h_{00} is replaced by ϕ_g

- In Newton's theory, $\phi_g(x)$ is determined by solving Poisson's eqn

$$\boxed{\nabla^2 \phi_g = 4\pi G \rho_m(x)} \quad \leftrightarrow \quad \phi_g = \sum_i \frac{-GM_i}{|x - x_i|}$$

(ρ_m: mass density $= \sum_i m_i \delta(x - x_i)$)

This formula reproduces everything we want about the gravitational potential due to any mass distribution

$$\Rightarrow -\nabla^2 h_{00} = +2\nabla^2 \phi_g = 8\pi G \rho_m \simeq 8\pi G T^{00} \quad \because \quad T^{00} \sim \rho_{energy} \sim \rho_m c^2$$

& using units in which $c = 1$

It's not a good eqn, it has 00-component all around which means its a tensor, but it could be 00-comp. of a tensor eqn $X^{\mu\nu}(g, dg, d^2g) = 8\pi G T^{\mu\nu}$

ie. what we want is some tensor $X^{\mu\nu}$ constructed from $g_{\mu\nu}$ (metric) & its derivatives (upto 2nd) which reduces to $-\nabla^2 h_{00}$ in the Newtonian limit $|h_{\mu\nu}| << 1$

- when we write down eqn with more that two derivitives, very often has instabilities, Ostrogradkis' theorem ~~states~~ proved that if we have more than two derivitives in the eqn then atleast generically you have instabilities & clearly we need two derivitives because of dimensions & we have Riemann Tensor which is formed from the 2nd derivitive

problem is $R^{\lambda}_{\sigma\mu\nu}$ has 4 indices

antisymmetric under chang $\mu \leftrightarrow \nu$

we would like to write the field eqn's as $X^{\mu\nu} = 8\pi G T^{\mu\nu}$

what can I do to make it a symmetric tensor $X^{\mu\nu}$?

$$[\nabla_\mu, \nabla_\nu] V_\sigma = -R^{\lambda}_{\sigma\mu\nu} V_\lambda$$

R is antisymmetric under μ, ν, so cannot contract with g orelse we'll get zero

This is where Einstein went wrong, he wanted to write something in terms of Reimann Tensor

- what we can do is $R^{\lambda}{}_{\sigma\lambda\nu}(g, dg, d^2g) = R_{\sigma\nu}$ --- Ricci Tensor ; turns out to be symmeter under $\sigma \to \nu$

- Einstein's first attempt : $X^{\mu\nu} \propto R^{\mu\nu}$ was wrong

$X^{\mu\nu} = 8\pi G T^{\mu\nu}$; but $\nabla_\mu T^{\mu\nu} = 0$ --- this is the basic property. Equivalence principle says take the laws of $T^{\mu\nu}$ must satisfy in order to be flat spacetime physics $\partial_\mu T^{\mu\nu} = 0$ & stick it to curved spacetime. called stress-energy tensor

so to obtain a consistent eqn we must have $\nabla_\mu X^{\mu\nu} = 0$ as an identity

Can we make a tensor out of the Reimann Tensor which satisfies above eqn ?

Bianchi identity :

$$[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0 \quad \text{(Jacobi)}$$

$$\Big([\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]]\Big) V_\sigma = 0$$

(Bianchi) $R_{\sigma\beta\mu\nu;\kappa} + R_{\sigma\beta\nu\kappa;\mu} + R_{\sigma\beta\kappa\mu;\nu} = 0$

$,\nu = \partial_\nu$
$;\nu = \nabla_\nu$

So what we're goanna show is that straight from the Bianchi identity we can construct a covariantly constant tensor $X^{\mu\nu}$ ie. c.d. of it is zero (covariant divergence ; c.d.)

Take Bianchi & multiply it with $g^{\sigma\mu}$

- we can use $\nabla_\lambda g^{\sigma\mu} = 0$ - metricity which means that we're allowed to push g through derivitives freely

$$R_{\beta\nu;\kappa} + R^{\sigma}{}_{\beta\sigma\nu\kappa;\sigma} - R_{\beta\kappa;\nu} = 0$$

Multiply with $g_{\nu\beta}$,

$$R_{;\kappa} - \underbrace{R^{\nu\sigma}{}_{\nu\kappa;\sigma}}_{R^{\sigma}{}_{\kappa;\sigma}} - R^{\beta}{}_{\kappa;\beta} = 0 \qquad \because R_{\sigma\beta} = R^{\lambda}{}_{\sigma\lambda\beta} \text{ Ricci Tensor}$$

$R = g^{\sigma\beta} R_{\sigma\beta}$ Ricci scalar

$R_{\sigma\beta\nu\kappa} = -R_{\beta\sigma\nu\kappa}$ (proof later.)
antis antis

$$\left(-2R^{\sigma}_{\ k} + R g^{\sigma k}\right)_{;\sigma} = 0 \qquad G^{\sigma k}$$ Einstein Tensor

$$\left(R^{\sigma}_{\ k} - \frac{1}{2} g^{\sigma k} R\right)_{;\sigma} = 0 \Rightarrow \boxed{X^{\sigma k} \equiv R^{\sigma k} - \frac{1}{2} g^{\sigma k} R} \text{ obeys}$$

$$\nabla_{\sigma}(X^{\sigma k}) = 0$$

$\Rightarrow$ candidate eqn $AG^{\mu\nu} = 8\pi G T^{\mu\nu}$ ——— ⊛

A is constant fixed by checking correspondence with Newtonian limit

where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$,

$R^{\mu\nu} = R^{\lambda\mu}{}_{\lambda}{}^{\nu}$; $R = R^{\lambda\sigma}{}_{\lambda\sigma}$

Recap on Reimann Tensor:

$$R^{\sigma}_{\ \rho\mu\nu} = -\Gamma^{\sigma}_{\rho\mu,\nu} + \Gamma^{\sigma}_{\rho\mu}\Gamma^{\beta}_{\rho\nu}$$ antisymmetric under $\mu \leftrightarrow \nu$ because it was defined as a commutator of $\nabla_{\mu}, \nabla_{\nu}$

$$R^{\sigma}_{\ \rho\mu\nu}(0) \simeq \frac{1}{2}\eta^{\sigma\lambda}\left(g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu} - g_{\mu\lambda,\rho\nu} + g_{\rho\mu,\lambda\nu}\right)(0)$$

Symmetries:

$R_{\sigma\rho\mu\nu}$ – i) symmetric under exchange of 1st pair & 2nd pair of indices ($= R_{\mu\nu\sigma\rho}$)

ii) antisymmetric on 1st pair & 2nd pair of indices ($= -R_{\rho\sigma\mu\nu}$, $= -R_{\sigma\rho\nu\mu}$)

iii) cyclic in last 3 indices $R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0$

- Riemann Tensor has $\frac{D^2(D^2-1)}{12}$ ind. components in D-spacetime dim^{n}s.

we are now going to calculate $G^{\mu\nu}$ in the weak field limit ($g_{\mu\nu} \approx \eta_{\mu\nu} + h_{00}$) & match up with Poisson's eqn $\nabla^2\phi_g = 4\pi G \rho_m$ by identifying h_{00} with ϕ_g

- we would like to fix const. A by comparing with the Newtonian limit

First, consider the spatial components of ⊛

$|T^{ij}| \ll T^{00}$ ie. pressure is negligible compared to energy density, in for typical forms of matter like gas pressure depends on square of velocity of particle, so its same as saying $v \ll c$

$\Rightarrow G^{ij}$ is small i.e. $R^{ij} \approx \frac{1}{2}\delta^{ij} R$ ← Ricci scalar
↑ Ricci tensor ——— ①

But $R = g^{\mu\nu} R_{\mu\nu} \approx \eta^{\mu\nu} R_{\mu\nu} = -R_{00} + R_{ii}$
$\sim \partial^2 h_{\mu\nu}$
$g \sim \eta + h$ $= -R_{00} + \frac{3}{2}R$ --- from ①

$\Rightarrow R_{00} = \frac{1}{2}R \Rightarrow G_{00} \approx R_{00} + \frac{1}{2}R \approx 2R_{00}$

$\therefore R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(h_{\alpha\nu,\beta\mu} - h_{\beta\nu,\alpha\mu} - h_{\mu\alpha,\beta\nu} + h_{\beta\mu,\alpha\nu}\right) + O(h^2)$

Ricci Tensor, $R_{00} = \eta^{\alpha\mu} R_{\alpha 0\mu 0} = -R_{0000} + R_{i0i0}$ $\because \eta^{\alpha\mu} = \delta^{ij}$
$R_{0000} = 0$

we assumed 'h' has no time derivitives,

$R_{i0i0} = -\frac{1}{2} h_{00,ii} = -\frac{1}{2}\nabla^2 h_{00}$

$\Rightarrow R_{00} = -\frac{1}{2}\nabla^2 h_{00}$

Back to eq^n ⊛ $A G^{00} \approx 2AR_{00} = -A\nabla^2 h_{00}$

$8\pi G T^{00} = 8\pi G \rho_m$; $h_{00} = -2\Phi_g$

$A \cdot \nabla^2 \Phi_g = 4\pi G \rho_m$ is correct for $A = 1$

curvature ↓ energy ↓

So we have shown that $\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}}$ ⇒ Poisson eq^n

Einstein field eq^n ↑

dimens of $G_{\mu\nu} = L^{-2}$ (two derivitives of metric) ; metric is dimensionless
$T^{\mu\nu}$: energy

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

* Gauge theories & gravitational waves :

Analogy with Maxwell's theory -

Maxwell EOM's are linear but not Einstein's, so its solutions are much more richer

- In Maxwell's set $J^\nu = 0$, $\partial_\mu F^{\mu\nu} = 0$ gives electromagnetic waves
ie. E & B propagates as waves at the speed of light ;
in GR, its the metric & perturbation of metric that will go at speed of light

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

A_μ is analogous to $g_{\mu\nu}$ because $g_{\mu\nu}$ is something that changes under coordinate transformations, same is with A_μ is vector potential in some gauge ; their exact form should never be taken too seriously

gauge transf^n : $A_\mu \to A_\mu + \partial_\mu \chi$

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{dx^\alpha dx^\beta}{dx'^\mu dx'^\nu} g_{\alpha\beta}$$

only this part is wave eq^n

we get $\partial_\mu \partial^\mu A_\nu - \partial_\nu (\partial_\mu A^\mu) = 0$ ---- ① ---- doesn't look like wave eq^n

↑ differential eq^n

Its a linear eq^n so Fourier transform is always a good idea

$$A_\mu = \sum_k \tilde{A}^{(k)}_\mu e^{ik.x} \qquad k.x = k^\nu x_\nu = -k^0 x^0 + \underline{k}.\underline{x}$$

$$\Rightarrow (k^2 \delta^\mu_\nu - k^\mu k_\nu)\tilde{A}^\nu(k) = 0 \quad \leftarrow \text{algebraic eq}^n \text{ —— ②}$$

↑ matrix operator

ie. we have some matrix times $\tilde{A}(k)$ to be zero $\Rightarrow$ matrix has zero eigenvalues ; infact $\tilde{A}(k)$ is zero eigenvector of this matrix

what will be a zero eigenvector of $(k^2\delta^\mu_\nu - k^\mu k_\nu)$ that is expressed in terms of k ? $\longrightarrow$ $(k^2\delta^\mu_\nu - k^\mu k_\nu)k^\nu = 0$

$\Rightarrow \tilde{A}^\nu(k) \longrightarrow \tilde{A}^\nu(k) + i\tilde{\varepsilon}(k)k^\nu$ is still a solution for all ε ($\tilde{\varepsilon}$ ← fourier transform of θ)

and that corresponds to the gauge freedom

$$A_\nu \longrightarrow A_\nu + \partial_\nu \theta$$

To solve Maxwell, choose θ (or ε) to set $\partial_\mu A^\mu = 0 \leftrightarrow k_\mu \tilde{A}^\mu = 0$, then eq^n ① or ② are solved if $k^2 = 0 \Rightarrow k^{0^2} - \underline{k}^2 = 0 \Rightarrow k^0 = \pm|\underline{k}|$

∴ Gauge fixing :

1) set $\partial_\mu A^\mu = 0$ Lorentz gauge

2) Residual gauge freedom

$A_\mu \to A_\mu + \partial_\mu \theta$ with $\partial^2 \theta = 0$

ie. even if you use up the gauge freedom to set $\partial_\mu A^\mu = 0$, there's residual gauge freedom because we can always shift A_μ by something proportional to k, that doesn't affect $\partial_\mu A^\mu = 0$ because $k^2 = 0$

$\therefore \tilde{A}_\mu \to \tilde{A}_\mu + iE(k)k_\mu$ with $k^2=0$; so gauge freedom actually allows you to set two things to zero ; we've got four-components of A^μ, one of them is set zero by $\partial_\mu A^\mu = 0$ and the residual gauge freedom allows you to set another component to be zero, the component along the k vector & that's how you get ~~from~~ to two transverse mode from four (propagating/polarizatⁿ)

So for EM, A_μ ; $4 = 2 + 1 + 1$
- 2 → physical propagating modes
- 1 → residual gauge freedom
- 1 → fixed by Lorentz gauge

and in GR, the metric has 10 components, 4 will be fixed by coordinate transformatⁿ, 4 ~~wi~~ by residual gauge freedom & we'll be left with 2 gauge fixed DOF like electromagnetic waves.

GR ; first problem is to fix a gauge

- Gauge freedom in electrodynamics is the choice that we make to remove the freedom $A \to A + \partial$ because the physical quantity is the field strength

In electromagnetism, one of the nicest gauge is $\partial_\mu A^\mu = 0$, its not invariant under $A \to A + \partial\theta$ because $\partial_\mu A^\mu$ then changes by $\partial^2\theta$

- so we want to write down an eqⁿ which is not Gauge invariant in order to fix the gauge. what do we have in GR which is not a tensor eqⁿ in order to somewhat fix the coordinates ?

we have $g'_{\mu\nu}(x') = g_{\alpha\beta}\frac{dx'^\alpha}{dx'^\mu}\frac{dx^\beta}{dx'^\nu}$

$g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$

$g'_{\mu\nu} \approx \eta_{\mu\nu} + h'_{\mu\nu}$

$x'^\mu \approx x^\mu + \varepsilon^\mu(x)$

$\Rightarrow h'_{\mu\nu} = h_{\mu\nu} - \varepsilon_{\mu,\nu} - \varepsilon_{\nu,\mu}$

ie. we can make small coordinate transformations which will change the metric from $h_{\mu\nu}$ to $h'_{\mu\nu}$ related by above formula

This is analogous to $A_\mu \to A_\mu + \partial_\mu \chi$ — A has single vector index, χ is a scalar

and $h_{\mu\nu}$ has two indices & $\varepsilon^\mu(x)$ is a vector

ie. instead of this single gauge DOF (χ), we have four of them, one gauge DOF for each μ ; $\varepsilon_\mu \leftrightarrow \chi$ or θ in E.M with 4 DOF in G.T.

- Now we have to write 4 eq^n to fix 4 DOF ; we have $\Gamma^\lambda_{\mu\nu}$ (4 × 10) — we'll have 40 components

Harmonic gauge $\longrightarrow$ $\boxed{g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0}$ — 4 eq^n which are not coordinate invariant & thus fix the gauge

if we work out Laplacian on a scalar

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{\mu\nu}\nabla_\mu\partial_\nu\phi = g^{\mu\nu}(\partial_\mu\partial_\nu\phi - \Gamma^\lambda_{\nu\mu}\partial_\lambda\phi)$$

c.d. of a scalar is just a normal derivitive

$$\Box\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi$$ in harmonic gauge 2nd term vanishes

in particular, if we regard the coordinates x^μ as four scalars (x^μ doesn't transform like a tensor, ∂x^μ does)

4D-wave equation $\to \boxed{\Box x^\alpha = g^{\mu\nu}\partial_\mu\partial_\nu x^\alpha = 0}$ $\quad \because \partial_\nu x^\alpha = \delta^\alpha_\nu$, $\partial_\mu \delta^\alpha_\nu = 0$

Linearised Einstein eq^n : $G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \Rightarrow R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$

Multiplying it by $g^{\mu\nu}$, $R = 2R$ $\quad \because g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4$

$\Rightarrow R = 0$

$\Rightarrow \boxed{R_{\mu\nu} = 0}$ — Einstein eq^n for $T^{\mu\nu} = 0$

ie. if the Einstein tensor is zero then Ricci scalar is zero which follows Ricci tensor to be zero

for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$R_{\mu\nu} = \frac{1}{2}\left(h^\lambda_{\nu,\mu\lambda} - \underbrace{h^\lambda_{\mu\nu,\lambda}}_{\Box h_{\mu\nu}} - h^\lambda_{\lambda,\mu\nu} + h^\lambda_{\mu,\nu\lambda}\right)$$

$$= -\frac{1}{2}\left(\Box h_{\mu\nu} - \underbrace{h^\lambda_{\nu,\mu\lambda} - h^\lambda_{\mu,\nu\lambda} + h^\lambda_{\lambda,\mu\nu}}_{=0 \text{ in harmonic gauge}}\right)$$

harmonic gauge is just $h^\lambda_{\nu,\lambda} - \frac{1}{2}h^\lambda_{\lambda,\nu} = 0$

for linearised theory is called "Transverse Trace Reversed" gauge (TT gauge)

consider

$h^{\lambda}_{\nu} - \frac{1}{2} h^{\alpha}_{\alpha} \delta^{\lambda}_{\nu}$, h^{α}_{α} is the trace of h^{λ}_{ν}

Taking trace of this quantity :

$h^{\beta}_{\beta} - 2h^{\alpha}_{\alpha} = -h^{\alpha}_{\alpha}$ ∵ Trace of identity matrix is 4

~~ie. Trace of the above quantity is minus the trace of h^{λ}_{ν}~~

~~"Trace reversal"~~

Transverse is when four-divergence is zero, $\partial_{\lambda}\underbrace{(h^{\lambda}_{\nu} - \frac{1}{2} h^{\alpha}_{\alpha} \delta^{\lambda}_{\nu})}_{\text{Trace reversal}} = 0$

∴ In TT gauge,

$$R_{\mu\nu} = -\frac{1}{2} \Box h_{\mu\nu}$$

↗ $\Box X^{\alpha} = 0$ for some ~~co~~ C.T. $X^{\alpha} = x^{\alpha} + \varepsilon^{\alpha}(x)$, $\Box x^{\alpha} = 0$ iff $\Box \varepsilon^{\alpha}(x) = 0$ ie. we pick the gauge in GR ($\Box x^{\alpha} = 0$) & we still have freedom ; its still harmonic if $\Box \varepsilon^{\alpha} = 0$

ie. we impose 4 conditions, we get to impose another 4 C.T's & that's how we go from 10 DOF of $h_{\mu\nu}$ to 2 by imposing 8 conditions

Recap ; Gravitational waves (which occur in weak field approximat^{n})

$|h_{\mu\nu}| << 1$; $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Einstein eq^{n} in absence of matter $R_{\mu\nu} = 0$; curvature ~~Ricci tenson is zero~~

$= -\frac{1}{2} \Box h_{\mu\nu}$ ---in weak field apprm. in harmonic gauge ← ①

where $\Box = \eta_{\mu\nu} \partial^{\mu} \partial^{\nu}$

Two ways of writing harmonic gauge $g^{\mu\nu} \Gamma^{\lambda}_{\mu\nu} = 0$ or $\Box x^{\alpha} = 0$

Transverse → $\partial_{\mu}\underbrace{(h^{\mu}_{\nu} - \frac{1}{2} h^{\lambda}_{\lambda} \delta^{\mu}_{\nu})}_{\text{~~trace reversed~~ } h^{\mu}_{\nu}} = 0$

After performing Fourier transform on $\partial_{\mu} \rightarrow ik_{\mu}$ & its contract^{n} with h^{μ}_{ν} tells that h is $\perp^{qr}$ to k (propagat^{n} vector) meaning of divergence to be zero

Lets solve eq^n ① : $h_{\mu\nu}(t,\underline{x}) = \sum_{\underline{x}} \left(\varepsilon_{\mu\nu}(k) e^{ik\cdot x} + \text{complex conjugate}\right)$

satisfies eq^n ① if $k^2 = 0$

$k\cdot\underline{x} = -k^0 t + \underline{k}\cdot\underline{x}$

$\therefore k^2 = 0 \Rightarrow k^{0\,2} - \underline{k}^2 = 0 \Rightarrow k^0 = \pm|\underline{k}|$

without loss of generality, we can choose $k^0 = |\underline{k}|$

so we have sol^n of eq^n, for every vector k we have an arbitrary amplitude $\varepsilon_{\mu\nu}(k)$ & phase $e^{ik\cdot x}$ (determines the phase of the wave);
General sol^n representing propagation of gravitational waves

- But we must take gauge condition into account

ie. $k_\mu \varepsilon^{\mu\nu} - \frac{1}{2} k^\nu \varepsilon^\lambda{}_\lambda = 0 \quad \forall\, \underline{k}$ —— ②

where $k^\mu = (k^0, \underline{k}) = (k, \underline{k})$; $k_\mu = (-k, \underline{k})$

so superficially, we have 10 ~~no.~~ complex no.'s for every $\underline{k}$ and applying condition ② we're remained with 6 complex no.'s for each $\underline{k}$ and we still have residual coordinate freedom,

ie. if $x^\mu \to x'^\mu + \varepsilon^\mu(x)$, then $\Box x'^\mu = 0$ if $\Box \varepsilon^\mu = 0$

under a coordinate transformat^n $g'_{\mu\nu}(x) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$ ~~if $g_{\alpha\beta}$~~

if $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ & $g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}$ & $x'^\mu = x^\mu + \varepsilon^\mu(x)$

$\Leftrightarrow x^\mu = x'^\mu - \varepsilon^\mu(x)$

$x^\mu \approx x'^\mu - \varepsilon^\mu(x')$ to first order in ε^μ

$\frac{\partial x^\alpha}{\partial x'^\mu}$ acting on ε ↓ $\quad$ $\frac{\partial x^\beta}{\partial x'^\nu}$ on ε ↓

$\therefore \eta_{\mu\nu} + h'_{\mu\nu} \simeq \eta_{\mu\nu} - \varepsilon^\alpha_{,\mu}\delta^\beta_\nu \eta_{\alpha\beta} - \delta^\alpha_\mu \varepsilon^\beta_{,\nu}\eta_{\alpha\beta} = \eta_{\mu\nu} + h_{\mu\nu} - \varepsilon_{\nu,\mu} - \varepsilon_{\mu,\nu}$

(to linear order in ε & h)

$(\delta^\alpha_\mu - \varepsilon^\alpha_{,\mu})(\delta^\beta_\nu - \varepsilon^\beta_{,\nu})\eta_{\alpha\beta}$ ⟹ $\boxed{h'_{\mu\nu} = h_{\mu\nu} - \varepsilon_{\nu,\mu} - \varepsilon_{\mu,\nu}}$ —— ③

$\eta_{\alpha\beta}$ $\quad$ $\varepsilon^\alpha_{,\mu}\delta^\beta_\nu$ $\quad$ $\varepsilon^\beta_{,\nu}\delta^\alpha_\mu$

#

If ε^{μ} satisfies $\Box \varepsilon^{\mu} = 0$, we remain in harmonic gauge, can express ε^{μ} in terms of sol^{n} of wave eq^{n}

$$\varepsilon^{\mu}(x) = \sum_{k} \left(i \varepsilon^{\mu}(k) e^{-ik.x} + c.c \right)$$

where $k^{0} = |k| = k$ as before

substituting in ③

$$\Rightarrow \varepsilon'_{\mu\nu}(k) = \varepsilon_{\mu\nu} + k_{\mu} c_{\nu} + k_{\nu} c_{\mu}$$

↑ polarizat^{n} tensor of gravitational waves changes by

we still have 4 arbitrary constants (c_{μ}) to play with at every k which can be used to change the polarizat^{n} tensor of the gravitons to put it into more convenient or canonical form

- we can check that this change in ε preserves gauge condition :

$$k^{\mu} \varepsilon_{\mu\nu} - \frac{1}{2} k_{\nu} \varepsilon^{\nu}_{\mu} = 0$$

change is → $k^{\mu}(k_{\mu} c_{\nu} + k_{\nu} c_{\mu}) - \frac{1}{2} k_{\nu} \times 2 k^{\mu} c_{\mu} = 0 \quad ; k^{2} = 0$

Lets consider g-wave travelling in the z-direction $k^{\mu} = (k, 0, 0, k)$ $\quad k^{\mu} = (-k, 0, 0, k)$

$$k^{\mu} x^{\nu} \eta_{\mu\nu} = -k^{t} + k^{z}$$

we can always rotate z-axis into any other axis & it's a linear theory so ~~we d~~ most general sol^{m} is the superposition of this of kind of sol^{n}, so we don't loose anything by picking up one direction.

Gauge condit^{n} reads : $k^{\mu}(\varepsilon^{\mu}_{\mu\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\varepsilon^{\lambda}_{\lambda}) = 0$

$$k^{\mu}(\varepsilon_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\varepsilon^{\lambda}_{\lambda}) = 0$$

↑ component of G.C

$\nu = 0 \quad k(\varepsilon_{00} + \varepsilon_{30}) + \frac{1}{2}k(-\varepsilon_{00} + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) = 0$

$\nu = 1 \quad k(\varepsilon_{03} + \varepsilon_{33}) - \frac{1}{2}k(-\varepsilon_{00} + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) = 0$

$\nu = 2 \quad k(\varepsilon_{01} + \varepsilon_{31}) = 0$

$\nu = 3 \quad k(\varepsilon_{02} + \varepsilon_{32}) = 0$

Adding & substracting first two eq

$$\varepsilon_{01} = -\varepsilon_{31} \quad ; \quad \varepsilon_{03} = -\frac{1}{2}(\varepsilon_{00} + \varepsilon_{33})$$
$$\varepsilon_{02} = -\varepsilon_{32} \quad ; \quad \varepsilon_{22} = -\varepsilon_{11}$$

we have eliminated 4 components in terms of other 6 ; still have residual freedom $\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu$

$$\Rightarrow \varepsilon'_{00} = \varepsilon_{00} - 2kC_0 \quad ; \quad \varepsilon'_{13} = \varepsilon_{13} + kC_1$$
$$\varepsilon'_{23} = \varepsilon_{23} + kC_2 \quad ; \quad \varepsilon'_{33} = \varepsilon_{33} + 2kC_3$$

can choose C_μ to set

$$\varepsilon'_{00} = \varepsilon'_{13} = \varepsilon'_{23} = \varepsilon'_{33} = 0$$

Thus we get

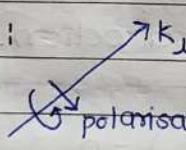
$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{21} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

this is transverse and traceless

Thus gravitational waves are transverse & traceless ; can always put linearised (keeping order h) gravitational wave in this form (superposition of such waves)

- instead of 10 complex no.'s $\varepsilon_{\mu\nu}$ we only have 2 $\Rightarrow$ g-waves have 2 ind. polarizat^n
- In E.M. $A^\mu \to (0, \varepsilon_1, \varepsilon_2, 0)$ for a wave along z-direct^n ; $\varepsilon_1, \varepsilon_2$ are two arbitary complex no.'s , we can say we have two polarisations one in x & other in y-direct^n.

Spin of graviton :



graviton in some direct^n k_μ ; the interesting parts of graviton are all transverse to the k-vector (after all the gauge fixing) and we want to do rotations which leave the momentum of the graviton unchanged because

$k^0 = |k| = k$
Energy

consider in 3D, we want to do rotations on k_μ but not change k-vector because k_μ is going in some particular direct^n & each k are ind. of each other ; "little group" SO(2) to do rotations leaving direct^n of k unchanges ; rotations around k

Transformat^n will be :

$$\varepsilon'_{ij} = O_{il} O_{jm} \varepsilon_{lm} \quad \cdots (*)$$

i.e. $\varepsilon' = O \varepsilon O^T$

In matrices,

$$\begin{pmatrix} \varepsilon'_{11} & \varepsilon'_{12} \\ \varepsilon'_{21} & \varepsilon'_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In our case little grp = rotations about z-axis

Trick from Group theory : Any abelian group (all elements commute) has only 1D irreducible representations

- $SO(2)$ is abelian, so all its irreps are 1D (1D representat^n grp : acts on something by a phase)

$$\left.\begin{aligned} \varepsilon_R &= \varepsilon_{11} - i\varepsilon_{12} = \varepsilon_+ - i\varepsilon_\times \\ \varepsilon_L &= \varepsilon_{11} + i\varepsilon_{12} = \varepsilon_+ + i\varepsilon_\times \end{aligned}\right\} \text{ transforms like } \begin{aligned} \varepsilon'_R &= e^{2i\theta} \varepsilon_R \\ \varepsilon'_L &= e^{-2i\theta} \varepsilon_L \end{aligned} \Rightarrow \text{spin-2}$$

(if something transforms as under rotations with twice the phase of a vector)

In E.M.

$$A_\mu \to (\varepsilon' \ \varepsilon'_2) = O \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \Rightarrow \begin{aligned} \varepsilon_+ &= \varepsilon_1 - i\varepsilon_2 \\ \varepsilon_- &= \varepsilon_1 + i\varepsilon_2 \end{aligned} \Rightarrow \begin{aligned} \varepsilon'_{R+} &= e^{i\theta} \varepsilon_+ \\ \varepsilon'_- &= e^{-i\theta} \varepsilon_- \end{aligned} \Big\} \text{photon spin-1}$$

- Gravitational radiation :

EM : $\partial_\mu F^{\mu\nu} = -J^\nu$ (source)

$\downarrow$

$\Box A^\nu$ radiat^n

time-varying J^ν leads to the emission of EM waves

GR : $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}$

Rewriting the eq^n by taking trace (multiplying by $g_{\mu\nu}$).

$$R - 2R = 8\pi G T^\nu_\mu \qquad \because g_{\mu\nu} g^{\nu\alpha} = \delta^\alpha_\mu = 4$$

$$R = -8\pi G T^\nu_\mu \qquad \Rightarrow g_{\mu\nu} g^{\mu\nu} = 4$$

$$\therefore R^{\mu\nu} = 8\pi G T^{\mu\nu} + \frac{1}{2} g^{\mu\nu} (-8\pi G T^\nu_\mu) = 8\pi G \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T^\lambda_\lambda \right)$$

(source)

$= -\Box h_{\mu\nu}$ (radiation) in TT gauge

$$h_{\mu\nu} = -4G \int \frac{d^3x'}{|\underline{x} - \underline{x}'|} T_{\mu\nu} \left(t - |\underline{x} - \underline{x}'|, \underline{x}' \right) \quad \cdots (*)$$

--- By Green's function

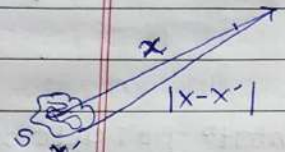
Retarted time $T^{\mu\nu}(t, \underline{x})$

ie we imagine that we've given a stress-energy tensor $T^{\mu\nu}$ & it obeys usual conservat^n laws

$\partial_\mu T^{\mu\nu} = 0$; its a funct^n of space & time

& we use it build source $s_{\mu\nu}$ and that will give metric perturbat^n at time t & position x.

$\therefore h_{\mu\nu}$ ⊛ represents the sol^n stating the fact that if we have a source that changes, it emits g-waves



quadrupole moment

$$Q_{kl} = \int d^3\underline{x}' \left(3x'^k x'^l - \delta^{kl}\underline{x}'^2\right) \rho(t, \underline{x}')$$

$\rho \to T^{00}(t, \underline{x}')$

Power radiated by a time-varying source is $P = \frac{G}{45} \dddot{Q}_{kl} \dddot{Q}_{kl} / c^5$

$\dddot{Q}_{kl} = \frac{d^3}{dt^3} Q_{kl}$

-- All this is Landau-Lifshitz formula ; confirmed by for binary pulsar in 1990s (Halsey + Taylor)

Black Holes !

Einstein eq^n with $T^{\mu\nu} = 0 \implies$ g-waves ; $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\implies R_{\mu\nu} \approx -\frac{1}{2}\Box h_{\mu\nu} = 0$$

we will solve $R_{\mu\nu} = 0$ nonlinearly (going to perturb metric by arbitrarily large amount) ; linearly (small perturbation)

and with a different symmetry

when we solved the eq^n linearly we assumed a symmetry ↦ we fourier transformed it & thought about each wave seperately, which only depended on $e^{-i\omega t + ikz} \approx h$ ie. wave is ind. of x & y, which is a translation symmetry (there's a 2D symmetry group of plane wave which are just translations in y or z-direct^n)

- This time we are not goanna use translations in 2D.

- we have a complex eqn & we want a localised soln for it, a blob ; what's the simplest kind of blob ? → sphere : spherical symmetry

How do we implement spherical symmetry ?

- line element $g_{\mu\nu} dx^\mu dx^\nu$ (is coordinate ind. quantity because the changing metric $g_{\mu\nu}(x)$ changes in just inverse way to dx^μ & dx^ν)

- four coordinates $t, \underline{x}$; $\underline{x} \to \underline{x}' = O\underline{x}$; $O_{ij} x_j$; O : rotation matrix
we want the line element to be invariant under rotations $SO(3)$

- we want quantities that are invariant under rotations :

dt , t , $r^2 = \underline{x}^2$, r , $r\,dr = \underline{x} \cdot d\underline{x} \Rightarrow dr$, $d\underline{x}^2 \neq dr^2$ *

$d\underline{x}^2 = dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$ ---- flat space metric in spherical polar coordinates

$d\Omega_2^2$ ← (metric on 2-sphere)

⇒ most general spherically symmetric line element is

$$ds^2 = -C(r,t)dt^2 + E(r,t)dr^2 - 2D(r,t)dt\,dr + F(r,t)r^2 d\Omega_2^2$$

(convinience)

must have i) $\det g \neq 0$ -- invertible ii) 1 -ve & 3 +ve eigenvalues

$$= (dt, dr, d\theta, d\phi) \begin{pmatrix} -C & -D & 0 & 0 \\ -D & +E & 0 & 0 \\ 0 & 0 & Fr^2 & 0 \\ 0 & 0 & 0 & \end{pmatrix} \begin{pmatrix} dt \\ dr \\ d\theta \\ d\phi \end{pmatrix}$$

Lets just replace $F(r,t)r^2 d\Omega_2^2$ by r'^2 ; we can write r' in terms (r,t) $\quad r' = \sqrt{F(r,t)}\, r$; $t' = t$

$$ds^2 = -c'(r',t')dt'^2 + E'(r',t')dr'^2 - 2D'(r',t')dt'dr' + r'^2 d\Omega_2^2$$

and then drop the primes since coordinates are our choice anyway

$$ds^2 = -c(r,t)dt^2 + E(r,t)dr^2 - 2D(r,t)dt\,dr + r^2 d\Omega_2^2$$

ugly term because off-diagonal in the metric

Can I remove $D(r,t)$ by a coordinate transformation? classmate Date Page

- we don't want to change r or mess up with $r^2 d\Omega_2^2$ term (Ω_2: 2-sphere)

ie. sphere is orbit of rotation group, if we take a point on sphere, apply a rotation, it carries us around the sphere; $r^2 d\Omega_2^2$ tells us that the area of the orbit of $SO(3)$ is $4\pi r^2$ → metric on the sphere, which is same as on flat space so the area is $4\pi r^2$: Basis of Schwarzchild coordinates that the area is always $4\pi r^2$

- we can still change t

Lets define t' by $dt' = f(r,t)\,(C(r,t)dt + D(r,t)dr)$ (arbitrary; integrating factor) → possible? ie. is there a funct^n $t'(r,t)$ s.t. $dt' \propto C dt + D dr$

$$ds^2 = \underbrace{-\frac{dt'^2}{f^2(r,t)C(r,t)}}_{-A(r,t)dt'^2} + \underbrace{\left(\frac{D^2(r,t)}{C(r,t)} + E(r,t)\right)}_{\text{arbitrary fun}^c\ B(r,t)} dr^2 + r^2 d\Omega_2^2 \quad \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

If t' exists, then $dt' = \frac{\partial t'}{\partial t}dt + \frac{\partial t'}{\partial r}dr = fC\,dt + fD\,dr$

$$\Rightarrow \underbrace{\frac{\partial t'}{\partial t} = f(r,t)C(r,t) \quad ; \quad \frac{\partial t'}{\partial r} = f(r,t)D(r,t)}$$

~~for consistency we must have~~ $\frac{\partial^2 t'}{\partial r \partial t} = \frac{\partial^2 t'}{\partial t \partial r}$ because p.d. commute

$$\Rightarrow \frac{\partial}{\partial r}\left(f(r,t)C(r,t)\right) = \frac{\partial}{\partial t}\left(f(r,t)D(r,t)\right) \quad \text{—— } (*)$$

If we don't have $f(r,t)$ ie. we set $f(r,t) = 1$ then there's no guarantee that $\frac{\partial}{\partial r}(C(r,t)) = \frac{\partial}{\partial t}(D(r,t))$

Thus $f(r,t)$ must a free funct^n to choose whatever we like from (*)

$$\partial_t f D + f \partial_t D = \partial_r f C + f \partial_r C \quad \text{—— } (\#)$$

can we always solve this eq^n? → Yes, atleast locally : imagine fixing $f(r,0) \Rightarrow \partial_t f(r,0)$

$\therefore \frac{\partial}{\partial t}(\#) \Rightarrow \frac{\partial^2 f}{\partial t^2}$ in terms of $f(r,0)$, $\frac{\partial F}{\partial t}(r,0)$ etc

$\therefore f(r,t) = f(r,0) + \partial_t f(r,t)\big|_{t=0} + \frac{1}{2}\partial_t^2 f(r,t)\big|_{t=0} + \cdots$

so, in general we can always solve eqn (#) to get atleast the Taylor series of $f(r,t)$ for all t.

$\Rightarrow$ left with $ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2 d\Omega_2^2$

dropping prime on t

- Schwarzchild was able to argue that the most general spherically symmetric metric can always be put in this form where its completely diagonal & there are only two arbitrary functn not fixed by spherical symmetry alone

Now, we will use Einstein eqn, $R_{\mu\nu} = 0$ to fix $A(r,t)$ & $B(r,t)$; we shall prove that most general spherically symmetric soln of $R_{\mu\nu} = 0$ is <u>static</u> ! (Birkhoff's theorem)

Notation : $\partial_t A = \dot{A}$; $\partial_r A = A'$ etc

$\Gamma^{\mu}_{\nu\lambda}$; $\Gamma^{t}_{tt} = \frac{\dot{A}}{2A}$

$= \frac{1}{2} g^{\mu\alpha}\left(g_{\lambda\alpha,\nu} + g_{\nu\alpha,\lambda} - g_{\nu\lambda,\alpha}\right)$

$\Gamma^{t}_{rr} = \frac{\dot{B}}{2A}$; $\Gamma^{t}_{tr} = \frac{\dot{A}}{2A}$, $\Gamma^{r}_{tt} = \frac{A'}{2B}$; $\Gamma^{r}_{tr} = \frac{\dot{B}}{2B}$; $\Gamma^{r}_{rr} = \frac{B'}{2B}$

$\Gamma^{r}_{\theta\theta} = -\frac{r}{B}$; $\Gamma^{r}_{\phi\phi} = -\frac{r\sin^2\theta}{B}$; $\Gamma^{\theta}_{r\theta} = \frac{1}{r}$; $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$

$\Gamma^{\phi}_{r\phi} = \frac{1}{r}$; $\Gamma^{\phi}_{\phi\theta} = \cot\theta$; all other components are zero

didn't see correctly

$-R^{\sigma}_{\lambda\mu\nu} \sim \partial\Gamma + \Gamma\Gamma$; $R_{\lambda\nu} = R^{\sigma}_{\lambda\sigma\nu}$:

$$R_{tt} = \frac{A''}{2B} - \frac{A'B'}{4B^2} + \frac{A'}{Br} - \frac{A'^2}{4AB} - \frac{\ddot{B}}{2B} + \frac{\dot{B}^2}{4B^2} + \frac{\dot{A}\dot{B}}{4AB} \quad \text{—— (i)}$$

$$R_{rr} = -\frac{A''}{2A} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} + \frac{B'}{Br} + \frac{\ddot{B}}{2A} - \frac{\dot{A}\dot{B}}{4A^2} - \frac{\dot{B}^2}{4AB} \quad \text{—(ii)}$$

$$R_{tr} = \frac{\dot{B}}{Br} \quad \text{—(iii)}$$

$$R_{\theta\theta} = 1 - \frac{1}{B} + \frac{rB'}{2B^2} - \frac{rA'}{2AB} \quad \text{—(iv)} \quad ; \quad R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \quad \text{—(v)}$$

All the components of Ricci tensor must be zero.

from (iii) $\Rightarrow \frac{\dot{B}}{Br} = 0$ ie. $\dot{B} = 0 \Rightarrow \frac{\partial}{\partial t} B(r,t) = 0 \Rightarrow B = B(r)$ only !

ie. Einstein eqn doesn't allow B to depend on time ; it should only depend on radius ; hint for static

consider (i) & (ii) ; To cancel the 2nd derivative term (i) + $\frac{A}{B}$(ii) we get,

$$\frac{1}{r}\left(\frac{A'}{B} + \frac{B'A}{B^2}\right) = 0 \Rightarrow BA' + B'A = 0 \Rightarrow (BA)' = 0 \Rightarrow BA = f(t)$$

$$\Rightarrow A = \frac{f(t)}{B(r)}$$

$$ds^2 = -\frac{f(t)\,dt^2}{B(r)} + B(r)dr^2 + r^2 d\Omega_2^2$$

To make it simpler, redefine t

f must be +ve, so define $dt' = \sqrt{f(t)}\,dt$ ie. $t' = \int^t \sqrt{f(t)}\,dt$

$$\Rightarrow ds^2 = -\frac{dt'^2}{B(r)} + B(r)dr^2 + r^2 d\Omega_2^2$$

---- metric is static

drop prime, now $A(r) = \frac{1}{B(r)}$

from (iv) : $1 - \frac{1}{B} + \frac{rB'}{2B^2} + \frac{rB'}{2B^2} = 0$ $\quad \because \frac{A'}{A} = -\frac{B'}{B}$ By differentiating $\frac{1}{B}$

$$1 - \frac{1}{B} + \frac{rB'}{B^2} = 0 \Rightarrow 1 = \left(\frac{r}{B}\right)'$$

By integrating $\frac{r}{B} = r + k$, k ; constant

$$\Rightarrow B = \frac{1}{1 + k/r}$$

$$ds^2 = -(1 + k/r)dt^2 + \frac{dr^2}{1 + k/r} + r^2 d\Omega_2^2$$

---- schwarzchild metric

what is k ?

for large r, $g_{\mu\nu} \approx \eta_{\mu\nu}$ because the correction $1 + \frac{k}{r}$ is much less

$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$ ... 3D flatspace in spherical polars

Minkowski

$\therefore$ for larger, $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$; $h_{00} = -\frac{k}{r} = -\frac{2\phi_g}{c^2} = \frac{2GM}{rc^2}$

ϕ_g ~~Newtonian~~ gravitational potential

$\Rightarrow ds^2 = -dt^2\left(1 - \frac{2GM}{rc^2}\right) + dr^2\left(1 - \frac{2GM}{rc^2}\right) + r^2 d\Omega_2^2$

M : mass as determined at large r by comparing to Newtonian gravity

& k is just related to M at large r

let $r_s = \frac{2GM}{rc^2}$ be the Schwarzchild radius $1 - \frac{2GM}{rc^2} = 1 - \frac{r_s}{r}$

- very strange thing happens as r decreases to r_s & becomes smaller than r_s ; $t \rightarrow$ spacial coordinates, $r \rightarrow$ time coordinate
- turns out that singularity in metric at $r = r_s$ is an artifact of the coordinates but singularity at $r = 0$ is genuine singularity & cannot be removed by changing coordinates

How do we prove that the singurality of this metric at $r = 0$ can never be removed by changing coordinates ?

→ Identify some quantity which doesn't change under any C.T. & show that if that quantity is infinite at $r = 0$ then there's nothing we can do ever to make it finite.

what quantities doesn't change under C.T. ? → scalars $\phi'(x') = \phi(x)$

we have Ricci scalar, but its zero everywhere already,

also Ricci tensor is zero. How do we make a scalar ?

Infact ~~Reimann~~ $\rightarrow R^{\lambda\sigma\nu\rho} R_{\lambda\sigma\nu\rho}$ = scalar = ? = (number)$\frac{(GM)^2}{r^6} = \infty$ at $r = 0$ in all coordinate systems (Real singularity)

~~Tensor cannot be zero for curved space~~

$\because R^{\lambda\sigma\nu\rho}$ is 2nd derivitive of metric & $\frac{1}{r^6}$ is its square

Trajectories of particles in the Schwarzchild metric :

Define $r_s = 2GM$

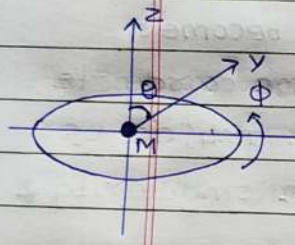
action for massive particle $= -m\int\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$ $\Rightarrow$ geodesic eqn & conservation laws

- action is obviously invariant under reperimetrizat^n of worldline, so λ can be any parameter along the worldline as long as it varies monotonically
- There's so much symmetry in sphere, that we don't need to use the geodesic eq^n (we can just use conservation laws)
- lets study the trajectory in propertime τ : $d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$

$$\Rightarrow \boxed{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1} \text{ ; where } \dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

one whose worldline is timelike ; can check that the geodesic eq^n is proportional to $\frac{d}{dt}(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)=0$ (implies)

- conservation laws :



$$S = -m\int\sqrt{\left(1-\frac{r_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{\left(1-r_s/r\right)}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2}\,d\lambda$$

$$= -\int d\lambda\, L(\lambda)$$

$\because \theta$ doesn't change with λ

convinient to put an object rotating around M in an orbit lying in x-y plane

$\therefore \theta = \pi/2$

symmetries :

$t = t + \text{const} \Rightarrow$ Energy

$\phi = \phi + \text{const} \Rightarrow$ angular momentum

$$\Rightarrow P_t \equiv \frac{\partial L}{\partial(dt/d\lambda)}, \quad P_\phi \equiv \frac{\partial L}{\partial(d\phi/d\lambda)} \quad \text{both conserved}$$

$$\therefore P_t = \frac{-M\left(1-r_s/r\right)\left(\frac{dt}{d\lambda}\right)}{\sqrt{\left(1-\frac{r_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{\left(1-r_s/r\right)}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2}} = -M\left(1-\frac{r_s}{r}\right)\left(\frac{dt}{d\tau}\right)$$

$$P_\phi = \frac{M(d\phi/d\lambda)r^2}{(1-r_s/r)\sqrt{\left(1-\frac{r_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{\left(1-\frac{r_s}{r}\right)}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2}} = Mr^2\left(\frac{d\phi}{d\tau}\right)$$

so by action & knowing the symmetries, we know what the conserved quantities

constraint $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$

$$\Rightarrow -\left(1-\frac{r_s}{r}\right)\dot{t}^2 + \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -1$$

Define $-\frac{p_t}{m} \equiv E$, $\frac{p_\phi}{m} \equiv J$

$$\Rightarrow \frac{-E^2}{(1-r_s/r)} + \frac{\dot{r}^2}{(1-r_s/r)} + \frac{J^2}{r^2} = -1 \quad \text{..... first-order ODE}$$

$$r^2\dot{\phi} = J \quad ; \quad \dot{t} = \frac{E}{1-r_s/r}$$

KE of particle $\longrightarrow \underbrace{\dot{r}^2 - \frac{r_s}{r} + \frac{J^2}{r^2}} - \frac{r_s J^2}{r^3} = E^2 - 1$

usual Newtonian terms

down by $\frac{r_s}{r}$ (dimensionless) compared to $\frac{J^2}{r^2}$

$\because \frac{r_s}{r} = \frac{2GM}{rc^2}$ Newtonian potential

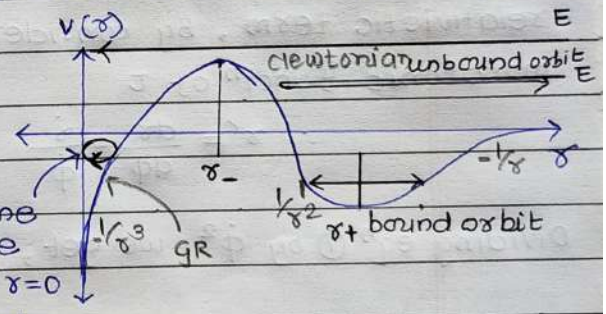
J^2/r^2 = centrifugal potential

$$\frac{r_s}{r} = \frac{2GM}{rc^2} \sim \frac{2v^2}{c^2} \ll 1$$

$\therefore v^2$ goes like $\frac{GM}{r}$ for Newtonian orbit. v circular

$\therefore \frac{r_s J^2}{r^3}$ is the GR correction for Newtonian, which is small as long as the planet is moving slower than the speed of light.

Let's Define effective Newtonian potential $V(r) = -\frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3}$



can come out of the singularity at $r=0$ & rebound back into the singularity

$F =$

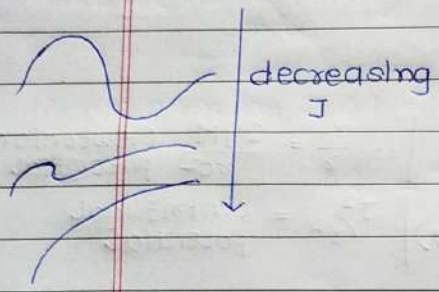
- There are two critical radii where $\frac{dV}{dr} = 0$; if there's no force on particle, then it can just sit at r_- or r_+ ie. particle can be in a circular orbit at r_- or r_+ (In Newtonian gravity there's only one circular orbit)
- If we perturb a particle at r_+ it's gonna stay stable, but pertubation of particle at r_- will take it to either $r=0$ or $r=\infty$; so r_+ is stable & r_- is unstable

$\frac{dV}{dr} = 0$ gives $\frac{r_s}{r} - \frac{2J^2}{r^3} - \frac{3r_s J^2}{r^4} = 0$

$\Rightarrow r_\pm = \frac{J^2}{r_s} \pm \sqrt{\left(\frac{J^2}{r_s}\right)^2 - 3J^2}$

$= \frac{J^2}{r_s}\left(1 \pm \sqrt{1 - \frac{3r_s}{J^2}}\right)$

As we decrease J to limit $\sqrt{3r_s}$, arguement of the squareroot vanishes & becomes imaginary, telling us the soln just disappears



so at limit $r_- = r_+ = 3r_s = \frac{6GM}{c^2}$ is called innermost stable circular orbit (Isco) below which one cannot orbit a BH.

- **Perihelian shift :** $\overbrace{\dot{r}^2 - \frac{r_s}{r} + \frac{J^2}{r^2}}^{\text{Newton}} - \frac{r_s J^2}{r^3} = \text{const}$ ——— ①

Also we have $\dot{\phi} = \frac{J}{r^2}$

As Newton had realised one can solve the eqn exactly without the relativistic term, by considering $r(\phi)$ instead $r(\tau)$, since ϕ is monotonic functn of τ

$\therefore r' = \frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}}$

Dividing eqn ① by $\dot{\phi}^2$ we get, $r'^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{\dot{r}^2 r^4}{J^2} = \left(\frac{r_s}{r} - \frac{J^2}{r^2}\left(1 - \frac{r_s}{r}\right)\frac{r^4}{J^2} + c\right)$

$= \left(\frac{r_s}{r} - \frac{J^2}{r^2}\left(1 - \frac{r_s}{r}\right) + \text{const}\right)\frac{r^4}{J^2}$

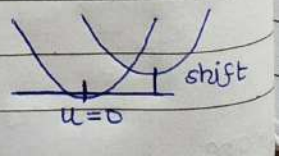
Define $u = \frac{1}{r}$; $u' = \frac{-r'}{r^2}$

$\Rightarrow \underbrace{u'^2 - \frac{r_s u}{J^2}}_{\text{Newton SHO}} + u^2 - \underbrace{r_s u^3}_{\text{relativistic (small) correction}} = \text{const}$ ——— ②

This is like an energy eqn, conservation of energy for u

for SHO we eliminate the linear term by shift of u, $u = \tilde{u} + a$

where $a = +\frac{1}{2}\frac{r_s}{J^2}$ cancels linear term in $\tilde{u}$



The soln is $u = \frac{r_s}{2J^2}(1+e\cos\phi)$ ← SHM with amplitude proportional to e

$\Rightarrow r = \frac{2J^2}{r_s}\frac{1}{(1+e\cos\phi)}$ ∴ elliptical orbit where elliptical e is any no. less than 1, $0<e<1$

- All Einstein had to do was to figure out the correction. To solve we'll treat $r_s u^3$ term as the perturbation

To get EOM of u, differentiate u w.r.t. ϕ, eqn ②

$$\underbrace{u'' = -\left(u - \frac{r_s}{2J^2}\right)}_{\text{shifted SHO with soln } (1+e\cos\phi)\frac{r_s}{2J^2}} + \underbrace{\frac{3}{2}r_s u^2}_{\text{GR perturbation}}$$

solves this

solve in perturbation theory: $u = u(0) + \delta u$, with small δu

$$\Rightarrow \delta u'' = -\delta u + \frac{3}{2}r_s u_0^2 = -\delta u + \underbrace{\frac{3}{2}r_s\left(\frac{r_s}{2J^2}\right)^2(1+e\cos\phi)^2}_{\text{This is a driven SHO}}$$

driving force

$(1+e\cos\phi)^2$

$1 + 2e\cos\phi + e^2\cos^2\phi = 1 + \underbrace{2e\cos\phi}_{\text{Resonant frequency !!!}} + e^2\left(\frac{\cos 2\phi + 1}{2}\right)$

what happens if we hang a pendulum which has a certain frequency and we drive it with a force of same frequency? Resonance

- so forget const, $\cos^2\phi$ terms, not interesting

$$\delta u'' = -\delta u + \frac{3}{4}\frac{r_s^3}{J^4}e\cos\phi$$

growing soln is $\delta u = \frac{3r_s^3}{8J^4}e\phi\sin\phi$

$$u \approx u(0) + \underbrace{\frac{3r_s^3}{8J^4}}_{\text{small coefficient}} e\phi\sin\phi$$

secularly growing term

perturbatn theory fails here

where $u(0) = \frac{r_s}{2J^2}(1+e\cos\phi)$

$$\approx \frac{r_s}{2J^2}\left(1+e\cos(1-\epsilon)\phi\right) + O(\epsilon^2)$$

where $\underbrace{\epsilon}_{\text{const}} = \frac{3r_s^2}{4J^2}$

resummation of perturb^n theory to prevent its breakdown

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